A canonical hyperkähler metric on the total space of a cotangent bundle

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Abstract

A canonical hyperkähler metric on the total space T^*M of a cotangent bundle to a complex manifold M has been constructed recently by the author in [K]. This paper presents the results of [K] in a streamlined and simplified form. The only new result is an explicit formula obtained for the case when M is an Hermitian symmetric space.

Introduction.

Constructing a hyperkähler metric on the total space T^*M of the cotangent bundle to a Kähler manifold M is an old problem, dating back to the very first examples of hyperkähler metrics given by E. Calabi in [C]. Since then, many people have obtained a lot of important results valid for manifolds M in this or that particular class (see, for example, the papers [DS], [BG], [Kr2], [Kr2], [N]). Finally, the general problem has been more or less solved a couple of years ago, independently by B. Feix [F] and by the author [K].

The metrics constructed in [F] and [K] are the same. In fact, this metric satisfies an additional condition which makes it essentially unique – which justifies the use of the term "canonical metric". But the approaches in [F] and [K] are very different. Feix's method is very geometric in nature; it is based on a direct description of the associated twistor space. The approach in [K] is much farther from geometric intuition. However, it seems to be more likely to lead to explicit formulas.

Unfortunately, the paper [K] is 100 pages long, and it is not written very well. Some of the proofs are not at all easy to understand and to check. The exposition is sometimes canonical to the point of obscurity.

Recently the author has been invited to give a talk on the results of [K] at the Second Quaternionic Meeting in Rome. It was a good opportunity to revisit the subject and to streamline and simplify some of the proofs. This paper, written for the Proceedings volume of the Rome conference, is an

attempt to present the results of [K] in a concrete and readable form. The exposition is parallel to [K] but the paper is mostly independent.

Compared to [K], the emphasis in the present paper has been shifted from canonical but abstract constructions to things more explicit and down-to-earth. The number of definitions is reduced to the necessary minimum. I have also tried to give a concrete geometric interpretation to everything that admits such an interpretation. In a sense, the exposition as compared to [K] intentionally goes to the other extreme. Thus the present paper is not so much a replacement for [K] but rather a companion paper – the same story told in a different way.

I should note that the canonical hyperkähler metric on T^*M has one important defect – namely, it is defined only in an open neighborhood $U \subset T^*M$ of the zero section $M \subset T^*M$. This raises a very interesting and difficult "convergence problem". One would like to describe the maximal open subset $U \subset T^*M$ where the canonical metric is defined, and to say when U is the whole total space T^*M. Unfortunately, very little is known about this. In fact, in the present paper we even restrict ourselves to giving the formal germ of the canonical metric near $M \subset T^*M$. The fact that this formal germ converges to an actual metric at least on an open subset $U \subset T^*M$ is proved in the last Section of [K]. The proof is long and tedious but completely straightforward. Since I do not know how to improve it, I have decided to omit it altogether to save space.

This reader will find the precise statements of all the results in Section 1. The last part of that Section contains a brief description of the rest of the paper, and indicates the parallel places in [K], the differences in notation and terminology and so on. The only thing in this paper which is completely new is the last Section 8. It contains an explicit formula for the canonical metric (or rather, for the canonical hypercomplex structure) in the case when M is an Hermitian symmetric space. The formula is similar to the general formula for symmetric M obtained by O. Biquard and P. Gauduchon in [BG].

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with O. Biquard and P. Gauduchon during my visit. In particular, the last Section is an attempt to compare [K] with the results in their beautiful paper [BG]. I am also grateful to M. Verbitsky, who read the first draft of the manuscript and suggested several improvements.

1 Statements and definitions.

To save space, we will assume some familiarity with hyperkähler and hypercomplex geometry. We only give a brief reminder. The reader will find excellent expositions of the subject in [B], [HKLR], [Sal1], [Sal2].

Let $\mathbb{H}$ be the algebra of quaternions. A smooth manifold X is called *almost quaternionic* if it is equipped with a smooth action of the algebra $\mathbb{H}$ on the tangent bundle TX. Equivalently, one can consider a smooth action on the cotangent bundle $\Lambda^1(M)$. To fix terminology, we will assume that $\mathbb{H}$ acts on $\Lambda^1(M)$ on the left.

An almost quaternionic manifold X is called *hypercomplex* if it admits a torsion-free connection preserving the $\mathbb{H}$-module structure on $\Lambda^1(M)$. Such a connection exists is unique. It is called the *Obata connection* of the hypercomplex manifold X.

A Riemannian almost quaternionic manifold X is called *hyper-hermitian* if the Riemannian pairing satisfies

$$(h\alpha_1, \alpha_2) = \left(\alpha_1, \overline{h}\alpha_2\right), \qquad \alpha_1, \alpha_2 \in \Lambda^1(M), h \in \mathbb{H},$$

where $\overline{h}$ is the quaternion conjugate to h. A hyper-hermitian almost quaternionic manifold X is called *hyperkähler* if the $\mathbb{H}$-action is parallel with respect to the Levi-Civita connection ∇_{LC}. In other words, X must be hypercomplex, and the Obata connection must be ∇_{LC}.

Let X be a almost quaternionic manifold. Every embedding $\mathbb{C} \hookrightarrow \mathbb{H}$ from the field of complex numbers to the algebra of the quaternions induces an almost complex structure on X. If X is hypercomplex, then all these induced almost complex structures are integrable. If X is also hyperkähler, then all these complex structures are Kähler with respect to the metric.

Throughout this paper it will be convenient to choose an embedding $I : \mathbb{C} \to \mathbb{H}$ and an additional element $j \in \mathbb{H}$ such that

$$\begin{aligned} j^2 &= -1 \\ j \cdot I(z) &= I(\overline{z}) \cdot j, \qquad z \in \mathbb{C} \end{aligned}$$

With these choices, every left $\mathbb{H}$-module $V_{\mathbb{R}}$ defines a complex vector space $V = V_I$ and a map $j : V \to \overline{V}$ which satisfies

$$j \circ \overline{j} = -\,\mathsf{id} \tag{1.1}$$

We will call V_I the *main complex structure* on the real vector space $V_{\mathbb{R}}$. Conversely, every pair $\langle V, j\rangle$ of a complex vector space V and a map $j : V \to \overline{V}$ which satisfies (1.1) defines an $\mathbb{H}$-module structure on the real vector space $V_{\mathbb{R}}$ underlying V.

The map $j : V \to \overline{V}$ can be considered as an automorphism $J : V_{\mathbb{R}} \to V_{\mathbb{R}}$ of the underlying real vector space. This map induces a second complex structure on $V_{\mathbb{R}}$. We will call it the *complementary* complex structure and denote the resulting complex vector space by V_J

Applying this to vector bundles, we see that an almost quaternionic manifold X is the same as an almost complex manifold X equipped with a smooth complex bundle map

$$j : \mathcal{T}(X) \to \overline{\mathcal{T}}(X) \tag{1.2}$$

from the tangent bundle $\mathcal{T}(X)$ to its complex-conjugate bundle $\overline{\mathcal{T}}(X)$ which satisfies (1.1).

It would be very convenient to have some way to know whether an almost quaternionic manifold X is hypercomplex or hyperkähler without working explicitly with torsion-free connections. For hypercomplex manifolds, the integrability condition is very simple. An almost quaternionic manifold X is hypercomplex if and only if both the main complex structure on X and the complementary almost complex structure X_J are integrable.

The simplest way to determine whether a hyper-hermitian almost quaternionic manifold X is hyperkähler is to consider the complex-bilinear form Ω on the tangent bundle $\mathcal{T}(X)$ given by

$$\Omega(\xi_1, \xi_2) = h(\xi_1, j(\xi_2)), \qquad \xi_1, \xi_2 \in \mathcal{T}(X), \tag{1.3}$$

where h is the Hermitian metric on X. Then (1.1) insures that the form Ω is skew-symmetric. The manifold X is hyperkähler if and only if both the $(2,0)$-form Ω and the Kähler ω are closed. Alternatively, one can define a $(2,0)$-form Ω_J for the complementary almost complex structure X_J instead of the $(2,0)$-form Ω for the main almost complex structure. Then X is hyperkähler if and only if both Ω and Ω_J are closed. Moreover, it suffices to require that these $(2,0)$-forms are holomorphic, each in its respective almost complex structure on X ("holomorphic" here means that $d\Omega$ is a form of type

$(3,0)$). Finally, if we already know that the manifold X is hypercomplex, then it suffices to require that only one of the forms Ω, Ω_J is a holomorphic 2-form.

If X is a Kähler manifold equipped with a closed $(2,0)$-form Ω, one can *define* a map $j : \mathcal{T}(X) \to \overline{\mathcal{T}}(X)$ by (1.3). Then X is hyperkähler if and only if this map j satisfies (1.1).

Consider now the case when $X = T^*M$ is the total space of the cotangent bundle to a Kähler manifold M with the kähler metric h. Then X carries a canonical holomorphic 2-form Ω. Moreover, the unitary group $U(1)$ acts on X by dilatations along the fibers of the projection $X \to M$, so that we have

$$z^*\Omega = z\Omega \tag{1.4}$$

for every $z \in U(1) \subset \mathbb{C}$. Using these data, we can formulate the main result of [K] as follows.

Theorem 1.1. *There exists a unique, up to fiber-wise automorphisms of X/M, $U(1)$-invariant Kähler metric h on $X = T^*M$, defined in the formal neighborhood of the zero section $M \subset T^*M = X$, such that*

(i) *h restricts to the given Kähler metric on the zero section $M \subset X$, and*

(ii) *the pair $\langle \Omega, h \rangle$ defines a hyperkähler structure on X near $M \subset X$.*

Moreover, if the Kähler metric h on M is real-analytic, then the formal canonical metric on X comes from a real-analytic hyperkähler metric defined in an open neighborhood of $M \subset X$.

The $U(1)$-action on T^*M is very important for this theorem. In fact, T^*M with this action belongs to a general class of hyperkähler manifolds equipped with a $U(1)$-action, introduced in [H], [HKLR].

Definition 1.2. We will say that a holomorphic $U(1)$-action on a hyperkähler manifold X is *compatible with the hyperkähler structure* if and only if

(i) the metric h on X is $U(1)$-invariant,

(ii) the holomorphic 2-form Ω satisfies (1.4),

(iii) the map $j : \mathcal{T}(X) \to \overline{\mathcal{T}}(X)$ satisfies

$$j(z^*\xi) = z \cdot z^*(j(\xi)), \qquad \xi \in \mathcal{T}(X), z \in U(1) \subset \mathbb{C}. \tag{1.5}$$

It is easy to check using (1.3) that every two of these conditions imply the third.

Theorem 1.1 can generalized to the following statement, somewhat analogous to the Darboux-Weinstein Theorem in symplectic geometry.

Theorem 1.3. *Let X be a hyperkähler manifold equipped with a regular compatible holomorphic $U(1)$-action. Then there exists an open neighborhood $U \subset X$ of the $U(1)$-fixed point subset $M = X^{U(1)} \subset X$ and a canonical embedding $\mathcal{L} : U \to T^*M$ such that the hyperkähler structure on U is induced by means of the map $\mathcal{L}$ from the canonical hyperkähler structure on T^*M.*

Here *regular* is a certain condition on the $U(1)$-action near the fixed points subset $X^{U(1)} \subset X$ which is formulated precisely in Definition 2.1 (roughly speaking, weights of the action on the tangent space $T_m X$ at every point $m \in X^{U(1)} \subset X$ should be 0 and 1). The map $\mathcal{L}$ will be called the *normalization map*. Note that this Theorem allows one to reformulate Theorem 1.1 so that the metric is indeed unique – not just unique up to a fiber-wise automorphism of X/M. To fix the metric, it suffices to require that the associated normalization map $\mathcal{L} : X \to X = \overline{T}M$ is identical. Metrics with this property will be called *normalized*.

There is also a form of Theorem 1.1 for hypercomplex manifolds (and it is this form which is the most important for [K] – all the other statements are obtained as its corollaries). To formulate it, we note that out of the three condition of Definition 1.2, the third one makes sense for almost quaternionic (in particular, hypercomplex) manifolds. We will say that a holomorphic $U(1)$-action on an almost quaternionic manifold X is *compatible with the quaternionic action* if Definition 1.2 (iii) is satisfied.

Let $\overline{T}M$ be the total space of the bundle $\overline{\mathcal{T}}(M)$ complex-conjugate to the tangent bundle of the manifold M. Then $\overline{T}M$ is a smooth manifold, and we have the canonical projection $\rho : \overline{T}M \to M$ and the zero section $i : M \to \overline{T}M$. The group $U(1)$ acts on $\overline{T}M$ by dilatations along the fibers of the projection ρ. Moreover, for any compatible hypercomplex structure on the $U(1)$-manifold $\overline{T}M$ the corresponding Obata connection ∇_O induces a torsion-free connection ∇ on M by the following rule

$$\nabla(\alpha) = i^*(\nabla_O \rho^* \alpha), \qquad \alpha \in \Lambda^1(M).$$

The hypercomplex version of Theorem 1.1 is the following.

Theorem 1.4. *Let M be a complex manifold M equipped with a holomorphic connection ∇ on the tangent bundle $\mathcal{T}(M)$ such that*

(i) ∇ *has no torsion, and*

(ii) *the curvature of the connection* ∇ *is of type* $(1,1)$.

Let $X = \overline{T}M$ *be the total space of the complex-conjugate to the tangent bundle* TM. *Let the group* $U(1)$ *act on* X *by dilatations along the fibers of the projection* $\rho : X \to M$.

Then there exists a unique, up to a fiber-wise automorphism of X/M, *hypercomplex structure on* X, *defined in the formal neighborhood of the zero section* $M \subset X$, *such that the embedding* $i : M \hookrightarrow X$ *and the projection* $\rho : X \to M$ *are holomorphic and the Obata connection on* X *induces the given connection* ∇ *on* M.

Moreover, if the connection ∇ *is real-analytic, then the hypercomplex structure on* X *is real-analytic in an open neighborhood* $U \subset X$ *of* $M \subset X$.

Note that *a priori* there is no natural complex structure on the space $X = \overline{T}M$ (we write $\overline{T}$ instead of T just to indicate the correct $U(1)$-action). Therefore Theorem 1.4 in fact claims two things: firstly, there exists an integrable almost complex structure on X, and secondly, there exists a map $j : \mathcal{T}(X) \to \overline{\mathcal{T}}(X)$ which extends it to a hypercomplex structure.

Connections ∇ that satisfy the conditions of this Theorem were called *kählerian* in [K] (see [K, 8.1.2]). I would like to thank D. Joyce for attracting my attention to his paper [J], where he uses the same class of connections to construct *commuting* almost complex structures on the total space TM of the tangent bundle to a complex manifold M. Joyce calls these connection *Kähler-flat*.

Theorem 1.4 also admits a generalized Darboux-like version in the spirit of Theorem 1.3 (see [K, Proposition 4.1]); we do not formulate it here to save space.

We will now give a brief outline of the remaining part of the paper. In Section 2 we consider an arbitrary $U(1)$-equivariant hypercomplex manifold X and construct the normalization map $\mathcal{L} : X \to \overline{T}M$, thus proving Theorem 1.3. This corresponds to [K, Section 4]. What we call *normalization* here was called *linearization* in [K]; *normalized* corresponds to *linear*. The terminology of [K] has been changed because it was misleading: connections on the fibration $\overline{T}M \to M$ that were called linear in [K] are not linear in the usual sense of the word.

Section 3 introduces $\mathbb{R}$-Hodge structures and the so-called Hodge bundles (Definition 3.1) which are the basis of our approach to $U(1)$-equivariant

quaternionic manifolds X. This corresponds to [K, Sections 2,3]. Proposition 3.3 is a version of [K, Proposition 3.1].

In Section 4 we turn to the case $X = \overline{T}M$ and introduce the so-called Hodge connections (Definition 4.2). This corresponds to [K, Section 5]. The proofs have been considerably shortened. There are also some new facts on the relation between our formalism and the objects one usually associates with hypercomplex manifolds. In particular, Lemma 4.3 is new.

Section 5 and Section 6 introduce the Weil algebra $\mathcal{B}^\bullet(M)$ of a complex manifold M (Definition 5.1) and then use it to prove Theorem 1.4. Section 5 contains the preliminaries; Section 6 gives the proof itself. This material corresponds to [K, Sections 6-8]. The approach has been changed in the following way. Keeping track of various gradings and bigradings and on the Weil algebra presents considerable difficulties: when the proof of Theorem 1.4 is written down, the number of indices becomes overwhelming. In [K] we have tried to handle this by an auxiliary technical device called *the total Weil algebra* ([K, 7.2.4]). It was quite a natural thing to do from the conceptual point of view. Unfortunately, the proof became more abstract than one would like. Here we have opted for the direct approach. To make things comprehensible, we rely on pictures (Figure 6.1, Figure 6.2) which graphically represent the Hodge diamonds of the relevant pieces of the Weil algebra $\mathcal{B}^\bullet(M)$.

Finally, Section 7 deals with things hyperkähler: we deduce Theorem 1.1 from Theorem 1.4. This corresponds to [K, Section 9]. We believe that the exposition has also been simplified, and the proofs are easier to check.

The last Section 8 of this paper is new. We try to illustrate our constructions by a concrete example of an Hermitian symmetric space M. We obtain a formula similar to [BG]. The last section of [K] contains the proof of convergence of our formal series in the case when the Kähler manifold M is real-analytic. In this paper, this proof is entirely omitted.

2 Normalization.

Of all the statement formulated in the last Section, the most straightforward one is the Darboux-like Theorem 1.3 and its hypercomplex version. In this Section, we explain how to construct the normalization map $\mathcal{L}$. Most of the proofs are replaced with references to [K].

We begin with some generalities. Assume that the group $U(1)$ acts smoothly on a smooth manifold X. For any point $x \in X$ fixed by the action, we have an action of $U(1)$ on the tangent space T_xX. Equivalently,

we have the weight decomposition

$$T_xX \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_k (T_xX)^k,$$

where $z \in U(1) \subset \mathbb{C}$ acts on $(T_xX)^k$ by multiplication by z^k. We will say that the fixed point x is *regular* if the only non-trivial weight components $(T_xX)^k$ correspond to weights $k = 0, 1$. The subset $X^{U(1)} \subset X$ of fixed points is a disjoint union of smooth submanifolds of different dimensions. Regular fixed points form a connected component of this set. Denote this component by $M \subset X$.

Let ϕ be the differential of the $U(1)$-action – that is, the vector field on X which gives the infinitesimal action of the generator $\frac{\partial}{\partial\theta}$ of the Lie algebra of the group $U(1)$. Assume further that X is a complex manifold and that $U(1)$ preserves the complex structure. Say that a point $x \in X$ is *stable* if for any $t \in \mathbb{R}$, $t \geq 0$ there exists $\exp(\sqrt{-1}t\phi)x$, and moreover, the limit

$$x_0 = \lim_{t \to +\infty} \exp(\sqrt{-1}t\phi)x$$

also exist. When the limit point x_0 does exist, it is obviously fixed by $U(1)$. Say that a stable point $x \in X$ is *regular stable* if the limit point is a regular fixed point, $x_\infty \in M \subset X$. Regular stable points form an open subset $X^{reg} \subset X$.

Definition 2.1. A complex $U(1)$-manifold X is called is *regular* if every point $x \in X$ is regular stable, $X^{reg} = X$.

For example, the total space of an arbitrary complex vector bundle on an arbitrary complex manifold is regular (if $U(1)$ acts by dilatations along the fibers). The submanifold of regular fixed points in this example is the zero section.

When the $U(1)$-manifold X is hypercomplex, we will say that it is regular if it is regular in the main complex structure. In this case, the subset $M \subset X$ of regular fixed points is a complex submanifold. Setting

$$\rho(x) = x_0 = \lim_{t \to +\infty} \exp(\sqrt{-1}t\phi)x$$

defines a $U(1)$-invariant projection $\rho : X \to M$.

Lemma 2.2 ([K, 4.2.1-3]). *The projection $\rho : X \to M$ is a holomorphic submersion.* $\square$

We can now define the normalization map. Consider the exact sequence

$$0 \longrightarrow \mathcal{T}(X/M) \longrightarrow \mathcal{T}(X) \longrightarrow \rho^*\mathcal{T}(M) \xrightarrow{d\rho} 0$$

of tangent bundles associated to the submersion $\rho : X \to M$. The differential ϕ of the $U(1)$-action is a vertical holomorphic vector field on X, $\phi \in \mathcal{T}(X/M)$. Applying the operator $j : \mathcal{T}(X) \to \overline{\mathcal{T}}(X)$ to ϕ gives a section of the bundle $\overline{\mathcal{T}}(X)$. We can project this section to obtain a section

$$d\rho(j(\phi)) \in \rho^*\overline{\mathcal{T}}(M)$$

of the pullback bundle $\rho^*\overline{\mathcal{T}}(M)$. But such a section tautologically defines a map $\mathcal{L} : X \to \overline{T}M$ from X to the total space of the complex bundle $\overline{T}$ on the manifold M. Since ϕ is $U(1)$-invariant, and j is of weight 1 with respect to the $U(1)$-action, the section $d\rho(j(\phi))$ is also of weight one. This means that the associated map

$$\mathcal{L} : X \to \overline{T}M$$

is $U(1)$-equivariant. We will call it the *normalization map* for the regular hypercomplex $U(1)$-manifold X.

Lemma 2.3 ([K, Proposition 4.1]). *The normalization map $\mathcal{L} : X \to \overline{T}M$ is an open embedding.* □

This Lemma essentially reduces Theorem 1.3 to Theorem 1.1.

A particular case occurs when $X = \overline{T}M$ is itself the total space of the complex-conjugate to the tangent bundle on a complex manifold M – or, more generally, an open $U(1)$-invariant neighborhood $U \subset \overline{T}M$ of the zero section $M \subset \overline{T}M$. As noted above, in this case the zero section $M \subset \overline{T}(M)$ coincides with the subset of fixed points. Therefore the normalization map $\mathcal{L} : U \to \overline{T}M$ is an open embedding from U into $\overline{T}M$, possibly different from the given one.

Definition 2.4. The hypercomplex structure on $U \subset \overline{T}M$ is called *normalized* if the normalization map $\mathcal{L} : U \to \overline{T}M$ coincides with the given embedding.

(In particular, when $U = \overline{T}M$ is the whole total space, the normalization map must be identical.)

To prove Theorem 1.1 and Theorem 1.4, it is sufficient to be able to classify all normalized hypercomplex structures on the $U(1)$-manifold $\overline{T}(M)$

and germs of such structures near the zero section $M \subset \overline{T}M$. It will be convenient to slightly rewrite the normalization condition. Namely, the identity map $\mathsf{id} : \overline{T}M \to \overline{T}M$ defines a section on $X = \overline{T}(M)$ of the pullback bundle $\rho^*\overline{T}(M)$. We will denote this section by τ and call it the *tautological section.* Then a hypercomplex structure on $U \subset X$ is normalized if and only if we have

$$j(\phi) = \tau \in \rho^*\overline{T}(M). \tag{2.1}$$

3 Hodge bundles.

The first step in the proof of Theorem 1.4 is to give a workable description of hypercomplex structures on the total space $X = \overline{T}M$. For this we use the language of $\mathbb{R}$-Hodge structures.

Recall that an $\mathbb{R}$-*Hodge structure* V *of weight* k is by definition a real vector space $V_{\mathbb{R}}$ equipped with a grading

$$V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_p V^{p,k-p} \tag{3.1}$$

such that

$$\overline{V^{p,q}} = V^{q,p}, \qquad p, q \in \mathbb{Z}, p + q = k. \tag{3.2}$$

Equivalently, instead of the grading (3.1) one can consider a $U(1)$ action on V defined by

$$z \cdot v = z^p V, \qquad v \in V^{p,q} \subset V, z \in U(1) \subset \mathbb{C}.$$

Then (3.2) becomes

$$\overline{z \cdot v} = z^k z \cdot \overline{v}, \qquad v \in V, z \in U(1) \subset \mathbb{C}.$$

When the weight k is equal to 1, this equation on the complex conjugation map becomes precisely (1.5). The difference between the complex conjugation map and the map j used to define quaternionic structures is that the first one is an involution, $\overline{\overline{v}} = v$, while for the second one we have $j(j(v)) = -v$. Nevertheless, we will exploit the similarity between them to describe quaternionic actions via Hodge structures. To do this, we use the following trick. Let V be an $\mathbb{R}$-Hodge structure of weight 1, and consider the map

$$\iota : V \to V$$

given by the action of $-1 \in U(1) \subset \mathbb{C}$ – in other words, let

$$\iota(v) = (-1)^p, \qquad v \in V^{p,1-p} \subset V.$$

Then the map

$$j = \iota \circ \overline{} : V \to \overline{V} \tag{3.3}$$

still satisfies (1.5), and (1.1) also holds. This turns V into a left $\mathbb{H}$-module. Conversely, every left $\mathbb{H}$ module $\langle V, j \rangle$ equipped with a $U(1)$-action on V such that j satisfies (1.5) defines an $\mathbb{R}$-Hodge structure of weight 1.

To use this for a description of hypercomplex structures on manifolds, we introduce the following.

Definition 3.1. Let X be a smooth manifold equipped with an action of the group $U(1)$, and let $\iota : X \to X$ be the action of the element $-1 \in U(1) \subset \mathbb{C}$. A *Hodge bundle $\mathcal{E}$ of weight k* on X is by definition a $U(1)$-equivariant complex vector bundle $\mathcal{E}$ equipped with a $U(1)$-equivariant isomorphism

$$\overline{} : \mathcal{E} \to \overline{\mathcal{E}}(k)$$

such that $\overline{} \circ \overline{} = \mathsf{id}$.

Here $\overline{\mathcal{E}}(k)$ is the bundle complex conjugate to $\mathcal{E}$, whose $U(1)$-equivariant structure is twisted by tensoring with the 1-dimensional representation $\mathbb{C}(k)$ of the group $U(1)$ of weight k,

$$z \cdot x = z^k x, \qquad x \in \mathbb{C}(k), z \in U(1) \subset \mathbb{C}.$$

When the $U(1)$-action on the manifold X is trivial, a weight-k Hodge bundle $\mathcal{E}$ on X is just the bundle of $\mathbb{R}$-Hodge structures of weight k in the obvious sense. In particular, if X is an almost complex manifold, then the bundle $\Lambda^k(X)$ of all complex-valued k-forms on X is a Hodge bundle of weight k.

When the $U(1)$-action on X is no longer trivial, every bundle $\mathcal{E}$ of $\mathbb{R}$-Hodge structures on X still defines a Hodge bundle. Thus $\Lambda^k(X)$ is still a weight-k Hodge bundle. But this Hodge bundle structure is not interesting, since it does not take into account the natural $U(1)$-action on $\Lambda^k(X)$. Assuming that the $U(1)$-action preserves the almost complex structure on X, we define instead a Hodge bundle structure on $\Lambda^1(X, \mathbb{C})$ by keeping the usual complex conjugation map and twisting the $U(1)$-action so that

$$\Lambda^1(X, \mathbb{C}) \cong \Lambda^{1,0}(X)(1) \oplus \Lambda^{0,1}(X)$$

as a $U(1)$-equivariant vector bundle. It is easy to check that this indeed defines on $\Lambda^1(X, \mathbb{C})$ a Hodge bundle structure of weight 1.

Assume now that the almost complex manifold X is equipped with an almost quaternionic structure which is compatible with the $U(1)$-action. Then the complex vector bundle $\Lambda^{0,1}(X)$ of $(0,1)$-forms on X already has a structure of a Hodge bundle of weight 1. The $U(1)$-action for this structure is the natural one, and the complex conjugation map is induced by the map $j : \mathcal{T}(X) \to \overline{\mathcal{T}}X$ via (3.3). This Hodge bundle structure completely determines the quaternionic action. More precisely, for every smooth $U(1)$-manifold X, every Hodge bundle $\mathcal{E}$ of weight 1 whose underlying real vector bundle $\mathcal{E}_{\mathbb{R}}$ is identified with the cotangent bundle $\Lambda^1(X, \mathbb{R})$ comes from a unique compatible almost quaternionic structure X.

The natural embedding $\Lambda^{0,1}(X) \subset \Lambda^1(X, \mathbb{C})$ is not a map of Hodge bundles – it is $U(1)$-equivariant, but it obviously does not commute with the complex conjugation map. This can be corrected. To do this, one has to look at the picture in a different way (which will turn out to be very useful). Return for a moment to linear algebra. Let $V_{\mathbb{R}}$ be a left $\mathbb{H}$-module, and let V, V_J be the complex vector spaces obtained from $V_{\mathbb{R}}$ by the main and the complementary complex structures. Consider the complex vector space $V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. This vector space does not depend on the $\mathbb{H}$-action on $V_{\mathbb{R}}$. Given an $\mathbb{H}$-action, we have the main and the complementary complex structure operators $I = I(\sqrt{-1}) : V_{\mathbb{R}} \to V_{\mathbb{R}}$ and $J : V_{\mathbb{R}} \to V_{\mathbb{R}}$ and the associated eigenspace decompositions

$$V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = V \oplus \overline{V}, \tag{3.4}$$

$$V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = V_J \oplus \overline{V_J} \tag{3.5}$$

Since the operators I and J anti-commute, these decompositions are distinct: we have

$$V \cap V_J = V \cap \overline{V_J} = \overline{V} \cap V_J = \overline{V} \cap \overline{V_J} = 0.$$

In particular, the composition

$$H : \overline{V} \to V_{\mathbb{R}} \otimes \mathbb{C} \to \overline{V_J} \tag{3.6}$$

of the canonical embedding in (3.4) and the canonical projection in (3.5) is an isomorphism. We will call it the *canonical isomorphism between the main and the complementary complex structures*. On the level of the real vector space $V_{\mathbb{R}}$, the map H is induced by a non-trivial automorphism $H : V_{\mathbb{R}} \to V_{\mathbb{R}}$ (in fact it is the action of the element $I(\sqrt{-1}) + j \in \mathbb{H}$). Conjugation with this map interchanges the operators I and J.

Return now to the case of an almost quaternionic manifold X. Then we claim that the complementary almost complex structure operator $J : \Lambda^1(X, \mathbb{C}) \to \Lambda^1(X, \mathbb{C})$ is a map of Hodge bundles. Indeed, it commutes with the complex conjugation map by definition. Therefore it suffices to show that it is $U(1)$-equivariant on $\Lambda^{0,1}(X) \subset \Lambda^1(X, \mathbb{C})$. But for every $v \in \Lambda^{1,0}(X)$ we have $J(v) = j(v) \in \Lambda^{0,1}(X)$, and the map j is of weight 1. Thus the operator J is indeed $U(1)$-equivariant (recall the twisting of the $U(1)$-action in the definition of the Hodge bundle structure on $\Lambda^1(X, \mathbb{C})$).

Since the endomorphism $J : \Lambda^1(X, \mathbb{C}) \to \Lambda^1(X, \mathbb{C})$ is a map of Hodge bundles, its eigenbundles

$$\Lambda^{1,0}_J(X), \Lambda^{0,1}_J(X) \subset \Lambda^1(X, \mathbb{C})$$

are Hodge subbundles. Therefore we obtain a canonical weight-1 Hodge bundle structure on the bundle $\Lambda^{0,1}_J(X)$ of $(0,1)$-forms for the *complementary* almost complex structure on X.

This Hodge bundle is not a new one. Indeed, the canonical isomorphism

$$H : \Lambda^{0,1}(X) \to \Lambda^{0,1}_J(X)$$

defined in (3.6) is $U(1)$-equivariant – it is obtained as a composition of $U(1)$-equivariant maps. Moreover, it is very easy to check that H commutes with the complex conjugation. Thus $\Lambda^{0,1}(X) \cong \Lambda^{0,1}_J(X)$ as Hodge bundles. But the projections

$$\Lambda^1(X, \mathbb{C}) \to \Lambda^{0,1}(X), \tag{3.7}$$

$$\Lambda^1(X, \mathbb{C}) \to \Lambda^{0,1}_J(X) \cong \Lambda^{0,1}(X), \tag{3.8}$$

are different. Only the second one is a Hodge bundle map.

All this linear algebra is somewhat tautological, but it becomes useful when we consider the integrability conditions on an almost quaternionic quaternionic structure. The real advantage of $\mathbb{R}$-Hodge structures over $\mathbb{H}$-modules is the presence of higher weights. Namely, the category of $\mathbb{H}$-modules admits no natural tensor product. On the other hand, the category of $\mathbb{R}$-Hodge structures and the category of Hodge bundles are obviously tensor categories. Thus, for example, the weight-1 Hodge bundle structure on the cotangent bundle $\Lambda^1(X, \mathbb{C})$ induces a weight-k Hodge bundle structure on the bundle $\Lambda^k(X, \mathbb{C})$ of k-forms.

To make use of these higher-weight Hodge bundles, we need a convenient notion of maps between Hodge bundles of different weights.

Definition 3.2. A bundle map (or, more generally, a differential operator) $f : \mathcal{E} \to \mathcal{F}$ between Hodge bundles $\mathcal{E}$, $\mathcal{F}$ of weights m, n is called *weakly Hodge* if it commutes with the complex conjugation map and admits a decomposition

$$f = \sum_{0 \leq p \leq n-m} f^p, \tag{3.9}$$

where $f^p : \mathcal{E} \to \mathcal{F}$ is of weight p with respect to the $U(1)$-action – in other words, f^p is $U(1)$-equivariant when considered as a map

$$f^p : \mathcal{E} \to \mathcal{F}(p).$$

We see that non-trivial weakly Hodge maps between Hodge bundles $\mathcal{E}$, $\mathcal{F}$ exist only when their weights satisfy $\mathsf{wt}\,\mathcal{F} \geq \mathsf{wt}\,\mathcal{E}$.

When the $U(1)$-action on the manifold X is trivial, the Hodge bundles $\mathcal{E}$ and $\mathcal{F}$ come from bundles of $\mathbb{R}$-Hodge structures on X, and the decomposition $f = \sum_p f^p$ of a weakly Hodge map $f : \mathcal{E} \to \mathcal{F}$ is simply the Hodge type decomposition,

$$f^p = f^{p,q}, \qquad p + q = \mathsf{wt}\,\mathcal{F} - \mathsf{wt}\,\mathcal{E}.$$

If the $U(1)$-action is not trivial, but preserves an almost complex structure on X, the de Rham differential $d = \partial + \bar{\partial} = d^{1,0} + d^{0,1} : \Lambda^0(X) \to \Lambda^1(X)$ is weakly Hodge. If the almost complex structure is integrable, then the same is true for the de Rham differential $d : \Lambda^k(X, \mathbb{C}) \to \Lambda^{k+1}(X, \mathbb{C})$ for every $k \geq 0$.

When the $U(1)$-manifold is almost quaternionic, we have a Hodge bundle structure of weight 1 on the bundle $\Lambda^{0,1}_J(X)$. Then the Dolbeault differential

$$D = \bar{\partial}_J : \Lambda^0(X, \mathbb{C}) \to \Lambda^{0,1}_J(X)$$

is weakly Hodge. Indeed, it is the composition of the weakly Hodge de Rham differential and the projection (3.8) which is a Hodge bundle map. Let

$$D = D^0 + D^1 \tag{3.10}$$

be the decomposition (3.9) for the weakly Hodge map D. Looking at the definition of the canonical isomorphism $H : \Lambda^{0,1}(X) \to \Lambda^{0,1}_J(X)$, we see that the component

$$D^0 : \Lambda^0(X, \mathbb{C}) \to \Lambda^{0,1}_J(X)$$

in the decomposition (3.10) coincides with the Dolbeault differential

$$\partial : \Lambda^0(X, \mathbb{C}) \to \Lambda^{0,1}(X) \cong \Lambda^{0,1}_J(X)$$

for the main complex structure.

Assume now that the almost quaternionic $U(1)$-manifold X is hypercomplex. The bundle $\Lambda^{0,k}_J(x)$ of $(0,k)$-forms on X_J is a Hodge bundle of weight k for every $k \geq 0$, and we have the Dolbeault differential

$$D = \bar{\partial}_J : \Lambda^{0,k}_J(x) \to \Lambda^{0,k+1}_J(x).$$

Since the projections $\Lambda^k(X, \mathbb{C}) \to \Lambda^{0,k}_J(X)$ are Hodge bundle maps for every $k \geq 0$, this Dolbeault differential is weakly Hodge. It turns out that this is a sufficient integrability condition for an almost quaternionic manifold equipped with a compatible $U(1)$-action.

Proposition 3.3. *Let X be an almost quaternionic manifold equipped with a compatible $U(1)$-action. Assume that the Dolbeault differential*

$$D : \Lambda^0(X, \mathbb{C}) \to \Lambda^{0,1}_J(X)$$

extends to a weakly Hodge derivation $D : \Lambda^{0,\bullet}_J(x) \to \Lambda^{0,\bullet+1}_J(X)$ of the algebra $\Lambda^{0,\bullet}_J(X)$ satisfying $D \circ D = 0$. Then the manifold X is hypercomplex.

Proof. It suffices to prove that both the main and the complementary almost complex structures on X are integrable. For this, it is enough to prove that the Dolbeault differentials

$$\begin{aligned} \bar{\partial}_J = D &: \Lambda^0(X, \mathbb{C}) \to \Lambda^{0,1}_J(X), \\ \bar{\partial} = D^0 &: \Lambda^0(X, \mathbb{C}) \to \Lambda^{0,1}_J(X) \cong \Lambda^{0,1}(X), \end{aligned}$$

extend to square-zero derivations of the exterior algebra $\Lambda^{0,\bullet}_J(X)$. The differential D extends by assumption. To extend D^0, take the component $D^0 : \Lambda^{0,\bullet}_J(x) \to \Lambda^{0,\bullet+1}_J(X)$ of the weakly Hodge map $D : \Lambda^{0,\bullet}_J(x) \to \Lambda^{0,\bullet+1}_J(X)$. Then $D^0 \circ D^0$ is a component in the decomposition (3.9) of the weakly Hodge map $D \circ D : \Lambda^{0,\bullet}_J(x) \to \Lambda^{0,\bullet+2}_J(X)$. Since $D \circ D = 0$, we also have $D^0 \circ D^0 = 0$. □

We will now say a couple of words about hyperkähler manifolds and Hodge bundles. Let X be an almost quaternionic $U(1)$-manifold. Then every Riemannian metric on X defines a $(2,0)$-form $\Omega_J \in \Lambda^{2,0}(X)$. It turns out that if the metric is hyper-hermitian and $U(1)$-invariant, then in terminology

of [K], the form Ω_J *of H-type* $(1,1)$. This means the following. Consider the form Ω_J as a bundle map

$$\Omega_J : \mathbb{R}(-1) \to \Lambda^{2,0}(X), \tag{3.11}$$

where $\mathbb{R}(-1)$ is the trivial bundle on X equipped with the so-called Hodge-Tate $\mathbb{R}$-Hodge structure of weight -1 – that is, the complex conjugation map on $\mathbb{R}(-1)$ is minus the complex conjugation map on $\mathbb{R}$, and the $U(1)$-equivariant structure is twisted by 1. The form Ω_J is said to be of H-type $(1,1)$ if the map (3.11) is a Hodge bundle map.

Conversely, every $(2,0)$-form $\Omega_J \in \Lambda^{2,0}(X)$ of H-type $(1,1)$ on an almost quaternionic $U(1)$-manifold X which satisfies a positivity condition

$$\Omega(\xi_1, I(\xi_2)) > 0, \qquad \xi_1, \xi_2 \in T(X, \mathbb{R}) \tag{3.12}$$

defines a $U(1)$-invariant hyper-hermitian metric on X. (See [K, 1.5.4], but the proof is almost trivial.)

If X is hypercomplex, then, as indicated in Section 1, the metric corresponding to such a form Ω_J is hyperkähler if and only if the form Ω_J is holomorphic, $D\Omega_J = 0 \in \Lambda^{2,1}_J(X)$.

Remark 3.4. In fact, using the $U(1)$-action on X, one can even drop the integrability condition. Indeed, if a form Ω_J of H-type $(1,1)$ on an almost quaternionic $U(1)$-manifold X satisfies $D\Omega_J = 0$, then it also must satisfy

$$D^0\Omega_J = D^1\Omega_J = 0.$$

The canonical endomorphism $H : \Lambda^1(X, \mathbb{R}) \to \Lambda^1(X, \mathbb{R})$, being the conjugation with a quaternion, preserves up to a coefficient the metric associated to Ω_J and interchanges the almost complex structure operators I and J. Therefore it sends Ω_J to a form proportional to Ω. Then $D^0\Omega_J = 0$ implies that not only Ω_J is holomorphic, but that Ω is holomorphic as well. This proves that X is hyperkähler (*a posteriori*, also hypercomplex). We will never need nor use this argument. An interested reader will find details in [K, 3.3.4].

4 Hodge connections.

We will now restrict our attention to the case when the $U(1)$-manifold X is the total space $\overline{T}M$ of the complex-conjugate to the tangent bundle of a complex manifold M. In this case, Proposition 3.3 is really useful, because

it turns out that the Hodge bundle algebra $\Lambda^{0,k}_J(X)$ does not depend on an almost quaternionic structure on X. To see this, denote by $\rho : X = \overline{T}M \to M$ the canonical projection, and let

$$\delta\rho : \rho^*\Lambda^1(M, \mathbb{C}) \to \Lambda^1(X, \mathbb{C})$$

be the codifferential of the map ρ. Then for every compatible hypercomplex structure on X, we have canonical bundle maps

$$\rho^*\Lambda^1(M, \mathbb{C}) \xrightarrow{\delta\rho} \Lambda^1(X, \mathbb{C}) \longrightarrow \Lambda^{0,1}_J(X)$$

Assume that the manifold X is a equipped with a hypercomplex structure satisfying the conditions of Theorem 1.4 – namely, assume that the projection $\rho : X \to M$ and the zero section $i : M \to X$ are holomorphic maps. Then the codifferential $\delta\rho$ is obviously a map of Hodge bundles. However, we also have the following.

Lemma 4.1 ([K, 5.1.9-10]). *The composition map*

$$\rho^*\Lambda^1(M, \mathbb{C}) \to \Lambda^{0,1}_J(X)$$

is an isomorphism of Hodge bundles in an open neighborhood of the zero section $M \subset X$. $\square$

From this point on, it will be convenient to only consider germs of hypercomplex structures defined near the zero section $M \subset X$. In other words, we replace $X = \overline{T}M$ with an unspecified and shrinkable $U(1)$-invariant open neighborhood of the zero section. Since we are only interested in hypercomplex structures on X that satisfy the conditions of Theorem 1.4, Lemma 4.1 shows that no matter what the particular hypercomplex structure on X is, we can *a priori* canonically identify the Hodge bundle $\Lambda^{0,1}_J(X)$ with the pullback bundle $\rho^*\Lambda^1(M, \mathbb{C})$. The only thing that depends on the hypercomplex structure is the derivation $D : \Lambda^0(X, \mathbb{C}) \to \Lambda^{0,1}_J(X) \cong \rho^*\Lambda^1(M, \mathbb{C})$.

To formalize the situation, we introduce the following.

Definition 4.2.

(i) A $\mathbb{C}$-*valued connection* Θ on X/M is a bundle map

$$\Theta : \Lambda^1(X, \mathbb{C}) \to \rho^*\Lambda^1(M, \mathbb{C})$$

which splits the codifferential $\delta\rho : \rho^*\Lambda^1(M, \mathbb{C}) \to \Lambda^1(X, \mathbb{C})$ of the projection $\rho : X \to M$.

(ii) The *derivation* $D : \Lambda^0(X,\mathbb{C}) \to \rho^*\Lambda^1(M,\mathbb{C})$ *associated* to a $\mathbb{C}$-valued connection Θ is the composition $D = \Theta \circ d$ of Θ with the de Rham differential d.

(iii) A *Hodge connection* Θ on X/M is a $\mathbb{C}$-valued connection such that the associated derivation $D : \Lambda^0(X,\mathbb{C}) \to \rho^*\Lambda^1(M,\mathbb{C})$ is weakly Hodge.

(iv) A Hodge connection Θ on X/M is called *flat* if the associated derivation D extends to a weakly Hodge derivation

$$D : \rho^*\Lambda^\bullet(M,\mathbb{C}) \to \rho^*\Lambda^{\bullet+1}(M,\mathbb{C})$$

of the pullback of the de Rham algebra $\Lambda^\bullet(M,\mathbb{C})$, and the extended map D satisfies $D \circ D = 0$.

Of course, a Hodge connection Θ is completely defined by the associated derivation $D : \Lambda^0(X,\mathbb{C}) \to \rho^*\Lambda^1(M,\mathbb{C})$. Conversely, an arbitrary weakly Hodge derivation

$$D : \Lambda^0(X,\mathbb{C}) \to \rho^*\Lambda^1(M,\mathbb{C})$$

comes from a Hodge connection if and only if we have

$$D\rho^* f = \rho^* df \in \rho^*\Lambda^1(M,\mathbb{C})$$

for every smooth function $f \in \Lambda^0(M,\mathbb{C})$. Say that a derivation

$$D : \Lambda^0(X,\mathbb{C}) \to \rho^*\Lambda^1(M,\mathbb{C})$$

is *holonomic* if the associated Hodge connection Θ induces an isomorphism

$$\Theta : \Lambda^1(X,\mathbb{R}) \to \rho^*\Lambda^1(M,\mathbb{C}) \tag{4.1}$$

between the real cotangent bundle $\Lambda^1(X,\mathbb{R})$ and the real bundle underlying the complex vector bundle $\rho^*\Lambda^1(M,\mathbb{C})$. Then Proposition 3.3 and Lemma 4.1 show that hypercomplex structures on X satisfying the conditions of Theorem 1.4 are in one-to-one correspondence with Hodge connections on X/M whose associated derivations are holonomic. Indeed, the isomorphism (4.1) induces a Hodge bundle structure of weight 1 on the cotangent bundle $\Lambda^1(X,\mathbb{R})$, hence an almost quaternionic structure on X. Applying Proposition 3.3, we see that flatness of the Hodge connection is equivalent to the integrability of this almost quaternionic structure. All derivations D that we will work with will be automatically holonomic – this will turn out to be a consequence of the normalization condition (2.1) (see Lemma 4.5).

The name "Hodge connection" invokes the notion of a connection on a smooth fibration. This is somewhat misleading. The problem is that a Hodge connection $\Theta : \Lambda^1(X, \mathbb{C}) \to \rho^*\Lambda^1(M, \mathbb{C})$ is only defined over $\mathbb{C}$. So it has a real part Θ_{Re} and an imaginary part Θ_{Im}. The real part

$$\Theta_{Re} : \Lambda^1(X, \mathbb{R}) \to \rho^*\Lambda^1(M, \mathbb{R})$$

is indeed a connection on the fibration $\rho : X \to M$ in the usual sense – that is, it defines a smooth splitting

$$\Lambda^1(X, \mathbb{R}) \cong \rho^*\Lambda^1(M, \mathbb{R}) \oplus \operatorname{Ker} \Theta_{Re}$$

of the real cotangent bundle $\Lambda^1(X, \mathbb{R})$ into a horizontal and a vertical part. The vertical part $\operatorname{Ker} \Theta_{Re}$ is canonically isomorphic to the relative cotangent bundle $\Lambda^1(X/M, \mathbb{R})$.

The imaginary part Θ_{Im}, on the other hand, vanishes on the subbundle $\rho^*\Lambda^1(M, \mathbb{R}) \subset \Lambda^1(X, \mathbb{R})$ and defines therefore a certain map

$$R_J : \Lambda^1(X/M, \mathbb{R}) \to \rho^*\Lambda^1(M, \mathbb{R}) \tag{4.2}$$

from the relative cotangent bundle $\Lambda^1(X/M, \mathbb{R})$ to the pullback bundle $\rho^*\Lambda^1(M)$.

Since X is an open subset in $\overline{T}M$, we can canonically identify the bundle $\Lambda^1(X/M)$ with the pullback bundle $\rho^*\Lambda^1(M)$. Under this identification, the map R_J becomes an endomorphism of the bundle $\rho^*\Lambda^1(M)$.

Typically, when a Hodge connection Θ comes from a hypercomplex structure on X, the associated real connection Θ_{Re} on X/M is *not* flat. It is only the sum

$$\Theta = \Theta_{Re} + \sqrt{-1}\Theta_{Im}$$

which is flat – but it is no longer a real connection. This situation is somewhat similar to what happens in C. Simpson's theory of Higgs bundles and harmonic metrics ([S]).

The presence of a non-trivial imaginary part Θ_{Im} seems to imply a contradiction. Indeed, a Hodge connection $\Theta : \Lambda^1(X, \mathbb{C}) \to \rho^*\Lambda^1(M, \mathbb{C})$ must by definition be compatible with the Hodge bundle structures – in particular, it must commute with the complex conjugation map. But this is different from "real". The reason for this is the twist by the involution $\iota : X \to X$ that we have introduced in the definition of a Hodge bundle. This can be seen clearly if instead of the connection Θ one considers the associated derivation D. The derivation $D : \Lambda^0(X, \mathbb{C}) \to \rho^*\Lambda^1(M, \mathbb{C})$ satisfies

$$D\iota^*\overline{f} = \iota^*\overline{Df} \tag{4.3}$$

for any $\mathbb{C}$-valued smooth function $f \in \Lambda^0(X, \mathbb{C})$. Take the decomposition $D = D_- + D_+$ into the odd and the even part with respect to the involution ι, so that every function $f \in \Lambda^0(X, \mathbb{C})$ we have

$$D_-(\iota^* f) = -\iota^* D_-(f), \qquad D_+(\iota^* f) = \iota^* D_+(f),$$

There is no reason for either one of these parts to vanish. But (4.3) shows that

$$D_- = \Theta_{Re} \circ d, \tag{4.4}$$

$$D_+ = \sqrt{-1}\Theta_{Im} \circ d, \tag{4.5}$$

where $d : \Lambda^0(X) \to \Lambda^1(X)$ is the de Rham differential.

The imaginary part Θ_{Im} of a Hodge connection Θ on X/M – or rather, the associated map R_J – by itself has a very direct geometric meaning in terms of the hypercomplex structure on X given by Θ. To describe it, consider the splitting

$$\Lambda^1(X, \mathbb{R}) = \Lambda^1(X/M, \mathbb{R}) \oplus \rho^* \Lambda^1(M, \mathbb{R}) \tag{4.6}$$

given by the real part Θ_{Re} and identify $\rho^* \Lambda^1(M, \mathbb{R}) \cong \Lambda^1(X/M, \mathbb{R})$.

Lemma 4.3. *The operator $j : \Lambda^1(X, \mathbb{R}) \to \Lambda^1(X, \mathbb{R})$ of the hypercomplex structure given by Θ can be written with respect to the decomposition* (4.6) *as the matrix*

$$\begin{pmatrix} 0 & -R_J^{-1} \\ R_J & 0 \end{pmatrix},$$

where $R_J : \rho^* \Lambda^1(M, \mathbb{R}) \to \rho^* \Lambda^1(M, \mathbb{R})$ is the bundle endomorphism (4.2).

Proof. Since $j^2 = -\mathsf{id}$, it suffices to prove that for every 1-form $\alpha \in \rho^* \Lambda^1(M, \mathbb{R})$ which lies in the horizontal part of (4.6), the 1-form $j(\alpha)$ is vertical, – that is,

$$\Theta_{Re}(j(\alpha)) = 0, \tag{4.7}$$

and moreover, that we have

$$\Theta_{Im}(j(\alpha)) = -\alpha. \tag{4.8}$$

Let α be such a form. By definition, the kernel $\operatorname{Ker} \Theta \subset \Lambda^1(X, \mathbb{C})$ of the projection $\Theta : \Lambda^1(X, \mathbb{C}) \to \rho^* \Lambda^1(M, \mathbb{C})$ is the subbundle $\Lambda^{1,0}_J(X)$ of $(1,0)$-forms for the complementary complex structure on X. Therefore we have

$$\alpha - \sqrt{-1} j(\alpha) \in \operatorname{Ker} \Theta,$$

which means that

$$\Theta(\alpha) = \sqrt{-1}\Theta(j(\alpha)).$$

Since $\alpha = \Theta(\alpha)$, equations (4.7) and (4.8) are the real and the imaginary parts of this equality. $\square$

In keeping with the general philosophy of this section, we will use the formula for j given by Lemma 4.4 to express the bundle endomorphism $R_J : \rho^*\Lambda^1(M, \mathbb{C}) \to \Lambda^1(M, \mathbb{C})$ entirely in terms of operators on the algebra $\rho^*\Lambda^\bullet(M, \mathbb{C})$. To do this, consider the tautological section of the pullback tangent bundle $\rho^*T(M)$, and let

$$\tau : \rho^*\Lambda^{\bullet+1}(M, \mathbb{C}) \to \rho^*\Lambda^\bullet(M, \mathbb{C})$$

be the operator given by contraction with this tautological section. Thus τ vanishes on functions, and for every 1-form $\alpha \in \Lambda^1(M, \mathbb{R})$ the function $\tau(\rho^*\alpha)$ is just α considered as a fiberwise-linear function on the total space $\overline{T}(M)$.

Lemma 4.4. *For any* 1*-form* $\alpha \in \Lambda^1(M, \mathbb{C})$ *we have*

$$R_J(\alpha) = -\sqrt{-1}D_+\tau(\alpha).$$

Proof. By (4.5), the right-hand side is equal to $\Theta_{Im}(d\tau(\alpha)) \in \rho^*\Lambda^1(M, \mathbb{C})$. Since the projection $\Theta_{Im} : \Lambda^1(X, \mathbb{C}) \to \rho^*\Lambda^1(M, \mathbb{C})$ vanishes on the subbundle $\rho^*\Lambda^1(M, \mathbb{C}) \subset \Lambda^1(X, \mathbb{C})$, this expression depends only the relative 1-form $P(d\tau(\alpha)) \in \Lambda^1(X/M, \mathbb{C})$ obtained from the 1-form $d\tau(\alpha) \in \Lambda^1(X, \mathbb{C})$ by the projection $P : \Lambda^1(X, \mathbb{C}) \to \Lambda^1(X/M, \mathbb{C})$. But $P(d\tau(\alpha))$ is precisely the image of the form $\alpha \in \rho^*\Lambda^1(M, \mathbb{C})$ under the canonical isomorphism $\rho^*\Lambda^1(M, \mathbb{C}) \cong \Lambda^1(X/M, \mathbb{C})$. $\square$

We will now rewrite in the same spirit the normalization condition (2.1) on the hypercomplex structure on X associated to D. For this we need to extends the canonical isomorphism $\Lambda^1(X/M, \mathbb{C}) \cong \rho^*\Lambda^1(M, \mathbb{C})$ to an algebra isomorphism $\Lambda^\bullet(X/M, \mathbb{C}) \cong \rho^*\Lambda^\bullet(M, \mathbb{C})$. Then the map

$$\tau : \rho^*\Lambda^{\bullet+1}(M, \mathbb{C}) \to \rho^*\Lambda^\bullet(M, \mathbb{C})$$

becomes the contraction with the relative Euler vector field (that is, the differential of the $\mathbb{R}^*$-action by dilatations along the fibers of the projection $\rho : X \to M$).

The normalization condition (2.1) involves a different vector field – namely, the differential ϕ of the standard action of the group $U(1)$. It will be

more convenient now to multiply it by $\sqrt{-1}$ (or, equivalently, to change the generator of the Lie algebra of the circle $U(1)$ from $\frac{\partial}{\partial\theta}$ to $z\frac{\partial}{\partial z}$). Denote by

$$\sigma : \rho^*\Lambda^{\bullet+1}(M,\mathbb{C}) \cong \Lambda^{\bullet+1}(X/M,\mathbb{C}) \longrightarrow \rho^*\Lambda^{\bullet}(M,\mathbb{C}) \cong \Lambda^{\bullet}(X/M,\mathbb{C})$$

the contraction with the vertical vector field $\sqrt{-1}\phi$. The operators σ and τ are related by

$$\sigma(\alpha) = \sqrt{-1}\tau(I\alpha), \qquad \alpha \in \Lambda^1(M,\mathbb{C}),$$

where $I : \Lambda^1(M,\mathbb{C}) \to \Lambda^1(M,\mathbb{C})$ is the complex structure operator – in other words,

$$\sqrt{-1}I = \begin{cases} -\mathsf{id} & \text{on} \quad \Lambda^{1,0}(M). \\ \mathsf{id} & \text{on} \quad \Lambda^{0,1}(M). \end{cases}$$

Lemma 4.5. *Let Θ be a Hodge connection on X/M, and let D_+ be the even component of the associated derivation $D : \Lambda^0(X,\mathbb{C}) \to \rho^*\Lambda^1(M,\mathbb{C})$.*

The hypercomplex structure on X given by Θ satisfies the normalization condition (2.1) *if and only for every* 1*-form $\alpha \in \Lambda^1(M,\mathbb{C})$ we have*

$$\sigma \circ D_+(f) = f.$$

where $f = \tau(\rho^*\alpha) \in \Lambda^0(X,\mathbb{C})$. Moreover, if this is the case, then the Hodge connection Θ is holonomic.

Proof. It suffices to check (2.1) by evaluating both sides on every 1-form $\alpha \in \rho^*\Lambda^1(M,\mathbb{C})$. Moreover, it is even enough to check it for forms of the type $\rho^*\alpha$, where $\alpha \in \Lambda^1(M,\mathbb{C})$ is a 1-form on M. Let α be such a form. We have to check that

$$j(\rho^*\alpha) \lrcorner \phi = \tau(\rho^*\alpha).$$

By Lemma 4.3 this is equivalent to

$$\sigma(R_J(\rho^*\alpha)) = -\sqrt{-1}\tau(\rho^*\alpha),$$

and by Lemma 4.4 this can be further rewritten as

$$-\sqrt{-1}\sigma(D_+\tau(\rho^*\alpha)) = -\sqrt{-1}\tau(\rho^*\alpha).$$

Replacing $\tau(\rho^*\alpha)$ with f gives precisely the first claim of the lemma.

To prove the second claim, we have to show that the map

$$\Theta : \Lambda^1(X,\mathbb{R}) \to \rho^*\Lambda^1(M,\mathbb{C})$$

is surjective. By definition, on the second term $\rho^*\Lambda^1(M,\mathbb{R}) \subset \Lambda^1(X,\mathbb{R})$ in the splitting (4.6) we have $\Theta = \Theta_{Re} = \mathsf{id}$. Therefore it suffices to prove that

$$\Theta_{Im} : \Lambda^1(X/M,\mathbb{R}) \to \sqrt{-1}\rho^*\Lambda^1(M,\mathbb{R})$$

is surjective. But by (4.5) and the first claim of the lemma, this is the inverse map to

$$\sigma : \sqrt{-1}\rho^*\Lambda^1(M,\mathbb{R}) \to S^1(M,\mathbb{R}) \cong \Lambda^1(X/M,\mathbb{R}). \qquad \square$$

5 The Weil algebra.

The final preliminary step in the proof of Theorem 1.4 is to reduce it from a question about the total space $X = \overline{T}M$ of the complex-conjugate to the tangent bundle on M to a question about the manifold M. To do this. we introduce the following.

Definition 5.1. The *Weil algebra* $\mathcal{B}^{\bullet}(M)$ of a complex manifold M is the algebra on M defined by

$$\mathcal{B}^{\bullet}(M) = \rho_*\rho^*\Lambda^{\bullet}(M,\mathbb{C}),$$

where $\rho : \overline{T}M \to M$ is the canonical projection.

This requires an explanation – indeed, for a vector bundle $\mathcal{E}$ on $\overline{T}M$, the direct image sheaf $\rho_*\mathcal{E}$ *a priori* is not a sheaf of sections of any vector bundle on M. We have to consider a smaller subsheaf. From now on and until Theorem 1.4 is proved, we will be interested not in hypercomplex structures on the total space $\overline{T}M$ but in their formal Taylor decompositions in the neighborhood of the zero section $M \subset \overline{T}M$. Therefore it will be sufficient for our purposes to define the direct image $\rho_*\mathcal{E}$ as the sheaf of sections of the bundle $\mathcal{E}$ on $\overline{T}M$ *which are polynomial along the fibers of the projection* $\rho : \overline{T}M \to M$. Formal germs of bundle maps on $\mathcal{E}$ will give formal series of maps between the corresponding direct image bundles.

Having said this, we can explicitly describe the Weil algebra $\mathcal{B}^{\bullet}(M)$. Our first remark is that $\mathcal{B}^k(M)$ is canonically a Hodge bundle on M of weight k. Moreover, since the $U(1)$-action on M is trivial, Hodge bundles on M are just bundles of $\mathbb{R}$-Hodge structures in the usual sense. Thus we have a Hodge type bigrading

$$\mathcal{B}^k(M) = \bigoplus_{p+q=k} \mathcal{B}^{p,q}(M)$$

and a canonical real structure on every one of the complex vector bundles $\mathcal{B}^k(M)$.

The projection formula show that for every k we have a canonical isomorphism

$$\mathcal{B}^k(M) \cong \mathcal{B}^0(M) \otimes \Lambda^k(M, \mathbb{C}).$$

These isomorphisms are compatible with the Hodge structures and with multiplication. The degree-0 Hodge bundle $\mathcal{B}^0(M)$ is a symmetric algebra freely generated by the bundle $S^1(M, \mathbb{C})$ of functions on $\overline{T}M$ linear along the fibers of $\rho : \overline{T}M \to M$. The complex vector bundle $S^1(M, \mathbb{C})$ is canonically isomorphic to the bundle $\Lambda^1(M, \mathbb{C})$ of 1-forms on M. However, the Hodge structures on these bundles are different. The Hodge type grading on $S^1(M, \mathbb{C})$ is given by

$$S^1(M, \mathbb{C}) = S^{1,-1}(M) \oplus S^{-1,1}(M, \mathbb{C}),$$

where $S^{1,-1}(M) \cong \Lambda^{1,0}(M)$ and $S^{-1,1}(M) \cong \Lambda^{0,1}(M)$ – the grading is the same as on $\Lambda^1(M, \mathbb{C})$ but graded pieces are assigned different weights. Moreover, the complex conjugation map on $S^1(M, \mathbb{C})$ is *minus* the complex conjugation map on $\Lambda^1(M, \mathbb{C})$. This is the last vestige of the twist by the involution $\iota : \overline{T}M \to \overline{T}M$ in Definition 3.1.

To simplify notation, denote by $S^k(M, \mathbb{C})$ the k-th symmetric power of the Hodge bundle $S^1(M, \mathbb{C})$. Then we have

$$\mathcal{B}^0(M) = \bigoplus_{k \geq 0} S^k(M, \mathbb{C}),$$

and the Weil algebra $\mathcal{B}^\bullet(M) = \mathcal{B}^0(M) \otimes \Lambda^\bullet(M, \mathbb{C})$ is the free graded-commutative algebra generated by $S^1(M, \mathbb{C})$ and $\Lambda^1(M, \mathbb{C})$ (where, contrary to notation, $S^1(M, \mathbb{C})$ is placed in degree 0).

It will be convenient to introduce another grading on the Weil algebra $\mathcal{B}^\bullet(M)$ by assigning to both of the generator bundles $S^1(M, \mathbb{C})$, $\Lambda^1(M, \mathbb{C})$ degree 1. We will call it *augmentation grading* and denote by lower indices, so that we have

$$\begin{aligned} S^1(M, \mathbb{C}) &= \mathcal{B}^0_1(M) \subset \mathcal{B}^0(M), \\ \Lambda^1(M, \mathbb{C}) &= \mathcal{B}^1_1(M) \subset \mathcal{B}^1(M). \end{aligned}$$

The augmentation grading corresponds to the Taylor decomposition near the zero section $M \subset \overline{T}M$. Namely, every formal germ near $M \subset \overline{T}M$ of a flat Hodge connection on $\overline{T}M/M$ induces a formal series

$$D = \sum_{k \geq 0} D_k \tag{5.1}$$

of algebra bundle derivations

$$D_k : \mathcal{B}^\bullet(M) \to \mathcal{B}^{\bullet+1}(M),$$

where each of the derivations D_k is weakly Hodge and has augmentation degree k. Their (formal) sum satisfies

$$D \circ D = 0.$$

Conversely, every formal series (5.1) induces a (formal germ of a) weakly Hodge derivation $D : \rho^*\Lambda^\bullet(M, \mathbb{C}) \to \rho^*\Lambda^{\bullet+1}(M, \mathbb{C})$ on the total space $\overline{T}M$. This derivation comes from a flat Hodge connection if and only if we have

$$D(f) = df$$

for every function $f \in \mathcal{B}^0_0(M) \cong \Lambda^0(M, \mathbb{C})$. Since we have $D \circ D = 0$, this immediately implies that D coincides with the de Rham differential d on the whole subalgebra

$$\Lambda^\bullet(M, \mathbb{C}) \subset \mathcal{B}^\bullet(M) \cong \mathcal{B}^0(M) \otimes \Lambda^\bullet(M, \mathbb{C}).$$

More precisely, we must have $D_1 = d$ on $\Lambda^\bullet(M, \mathbb{C})$, and all the other components $D_k, k \neq 1$ must vanish on this subalgebra. Since D is a derivation, this in turn implies that all the components $D_k : \mathcal{B}^\bullet(M) \to \mathcal{B}^{\bullet+1}(M)$ for $k \neq 1$ are not differential operators but bundle maps. Moreover, since the algebra $\mathcal{B}^\bullet(M)$ is freely generated by $S^1(M, \mathbb{C})$ and $\Lambda^1(M, \mathbb{C})$, and we know *a priori* the derivation D on the generator subbundle $\Lambda^1(M, \mathbb{C})$, it always suffices to specify the restriction $D : S^1(M, \mathbb{C}) \subset \mathcal{B}^0(M) \to \mathcal{B}^1(M)$.

The decomposition $D = D_- + D_+$ of a Hodge connection into an even and an odd part is quite transparent on the level of the Weil algebra – we simply have

$$D_- = \sum_{k \geq 0} D_{2k+1} \qquad\qquad D_+ = \sum_{k \geq 0} D_{2k}.$$

We will now rewrite the normalization condition (2.1) in terms of the Weil algebra. To do this, note that the map $\sigma : \rho^*\Lambda^{\bullet+1}(M, \mathbb{C}) \to \Lambda^\bullet(M, \mathbb{C})$ induces a bundle map $\sigma : \mathcal{B}^{\bullet+1}(M) \to \mathcal{B}^\bullet(M)$. This map is in fact a derivation of the Weil algebra. It vanishes on the generator bundle $S^1(M, \mathbb{C})$, while on the generator bundle $\Lambda^1(M, \mathbb{C})$ it is given by

$$\sigma = \begin{cases} \mathsf{id} : \Lambda^{1,0}(M) \to S^{1,-1}(M) \cong \Lambda^{1,0}(M), \\ -\,\mathsf{id} : \Lambda^{0,1}(M) \to S^{-1,1}(M) \cong \Lambda^{0,1}(M). \end{cases}$$

Then Lemma 4.5 immediately shows that the (formal germ of the) hypercomplex structure on $\overline{T}M$ induced by a derivation $D : \mathcal{B}^\bullet(M) \to \mathcal{B}^{\bullet+1}(M)$ is normalized if and only if we have

$$\sigma \circ D_+ = \mathsf{id} \tag{5.2}$$

on the generator subbundle $S^1(M, \mathbb{C}) \subset \mathcal{B}^0(M)$. It is convenient to modify this in the following way. Let $C : S^1(M, \mathbb{C}) \to \Lambda^1(M, \mathbb{C})$ be the isomorphism inverse to $\sigma : \Lambda^1(M, \mathbb{C}) \to \Lambda^1(M, \mathbb{C})$. Set $C = 0$ on the generator subbundle $\Lambda^1(M, \mathbb{C}) \subset \mathcal{B}^1(M)$ and extend it a derivation $C : \mathcal{B}^\bullet(M) \to \mathcal{B}^{\bullet+1}(M)$ of the Weil algebra. Both derivations C and σ are real. Moreover, the derivation C is weakly Hodge (the derivation σ is not – simply because it decreases the weight). Then the normalization condition is equivalent to

$$\begin{cases} D_0 = C, \\ \sigma \circ D_k = 0 \text{ on } S^1(M, \mathbb{C}) \subset \mathcal{B}^0(M) \text{ for every even } k = 2p \geq 1. \end{cases}$$

To sum up, formal germs near $M \subset \overline{T}M$ of normalized flat Hodge connections on $\overline{T}M$ are in a natural one-to-one correspondence with derivations

$$D = \sum_{k \geq 0} D_k : \mathcal{B}^\bullet(M) \to \mathcal{B}^{\bullet+1}(M)$$

of the Weil algebra $\mathcal{B}^\bullet(M)$ which satisfy the following conditions.

(i) $D \circ D = 0$.

(ii) $D_k : \mathcal{B}^\bullet(M) \to \mathcal{B}^{\bullet+1}(M)$ is a weakly Hodge algebra derivation of augmentation degree k.

(iii) $D_0 = C$.

(iv) $D_1 = d$ and $D_k = 0$, $k \neq 0$ on the subalgebra $\Lambda^\bullet(M, \mathbb{C}) \subset \mathcal{B}^\bullet(M)$.

(v) $\sigma \circ D_{2k} = 0$ on $S^1(M, \mathbb{C}) = \mathcal{B}^0_1(M) \subset \mathcal{B}^0(M)$ for every $k \geq 1$.

For every such derivation, the differential operator

$$D_1 : S^1(M, \mathbb{C}) = \mathcal{B}^0_1(M) \to \mathcal{B}^1_2(M) \cong S^1(M, \mathbb{C}) \otimes \Lambda^1(M, \mathbb{C})$$

satisfies the Leibnitz rule

$$D_1(fa) = fD_1(a) + adf, \qquad a \in S^1(M, \mathbb{C}), f \in \Lambda^0(M, \mathbb{C}).$$

Therefore it is a connection on the bundle $S^1(M, \mathbb{C})$. We postpone the proof of the following Lemma till the end of Section 7.

Lemma 5.2. *The connection D_1 on the bundle $S^1(M,\mathbb{C}) \cong \Lambda^1(M,\mathbb{C})$ coincides with the connection on M induced by the Obata connection for the hypercomplex structure on $\overline{T}M$ defined by the derivation D.*

With Lemma 5.2 in mind, we see that Theorem 1.4 is reduced to the following statement.

Proposition 5.3. *Let ∇ be a torsion-free connection on the cotangent bundle $\Lambda^1(M,\mathbb{C})$ of a complex manifold M. Assume that the curvature of the connection ∇ is of type $(1,1)$.*

Then there exists a unique derivation

$$D = \sum_{k\geq 0} D_k : \mathcal{B}^\bullet(M) \to \mathcal{B}^{\bullet+1}(M)$$

of the Weil algebra $\mathcal{B}^\bullet(M)$ of the manifold M such that D satisfies the conditions (i)-(v) *above and we have*

$$D_1 = \nabla$$

on $S^1(M,\mathbb{C}) \subset \mathcal{B}^0(M)$.

This ends the preliminaries. We now begin the proof of Proposition 5.3.

6 The proof of Proposition 5.3.

The proof proceeds by induction on the augmentation degree. Denote

$$D_{\leq k} = D_0 + D_1 + \cdots + D_k.$$

To base the induction, consider the derivation $D_{\leq 1}$. By assumptions it is equal to

$$D_{\leq 1} = C + D_1.$$

Let $R : \mathcal{B}^\bullet(M) \to \mathcal{B}^{\bullet+1}(M)$ be the composition

$$R = D_{\leq 1} \circ D_{\leq 1}.$$

Since the derivation $D_{\leq 1}$ of the graded-commutative algebra $\mathcal{B}^\bullet(M)$ is of odd degree, up to a coefficient the composition R coincides with the supercommutator $\{D_{\leq 1}, D_{\leq 1}\}$. In particular, it is also an algebra derivation.

The derivation $R : \mathcal{B}^\bullet(M) \to \mathcal{B}^{\bullet+2}(M)$ *a priori* has components R_0, R_1, R_2 of augmentation degrees 0, 1 and 2. However,

$$R_0 = \{C, C\} = 0.$$

Moreover,

$$R_1 = \{C, D_1\} : S^1(M, \mathbb{C}) \to \mathcal{B}^2_2(M) \cong \Lambda^2(M, \mathbb{C})$$

is precisely the torsion of the connection $\nabla = D_1$. Thus it vanishes by assumption. Since $C = 0$ on $\Lambda^\bullet(M, \mathbb{C}) \subset \mathcal{B}^\bullet(M)$, we also have $R_1 = 0$ on $\Lambda^1(M, \mathbb{C})$, which implies that $R_1 = 0$ everywhere. What remains is R_2. In general, it does *not* vanish, and to kill it we have to add new terms D_k.

We now turn to the induction step. Assume that for some $k \geq 2$ we are already given the derivation $D_{\leq k-1}$ which satisfies the conditions (ii)-(v) on page 27, and assume that the composition $D_{\leq k-1} \circ D_{\leq k-1}$ has no non-trivial components of augmentation degrees $< k$. Denote by $R_k : \mathcal{B}^\bullet(M) \to \mathcal{B}^{\bullet+1}(M)$ its component of augmentation degree k. We have to find a derivation $D_k : \mathcal{B}^\bullet(M) \to \mathcal{B}^{\bullet+1}$ which also satisfies (ii)-(v) and such that

$$(D_{\leq k-1} + D_k) \circ (D_{\leq k-1} + D_k) = 0$$

in augmentation degree k. This is equivalent to

$$\{D_0, D_k\} = \{C, D_k\} = -R_k. \tag{6.1}$$

The conditions (ii) and (iii) mean that $D_k : \mathcal{B}^\bullet(M) \to \mathcal{B}^{\bullet+1}(M)$ must be a weakly Hodge derivation which vanishes on $\Lambda^1(M, \mathbb{C}) = \mathcal{B}^1_1(M) \subset \mathcal{B}^1(M)$. The condition (iv) is only relevant for D_1. Finally, the condition (v) is relevant for all even k and means that

$$\sigma \circ D_k = 0$$

on $S^1(M, \mathbb{C}) \subset \mathcal{B}^0(M)$.

Because of (iii), it suffices to define D_k on the generator subbundle $S^1(M, \mathbb{C})$. Since D_k commutes with the complex conjugation map, it even suffices to consider only $S^{1,-1}(M) \subset S^1(M, \mathbb{C})$. Moreover, (iii) implies that it also suffices to check (6.1) only on $S^{1,-1}(M) = \mathcal{B}^{1,-1}_1(M) \subset \mathcal{B}^0(M)$. Note that on this subbundle we have $\{C, D_k\} = C \circ D_k$.

There will be two slightly different cases. The first is one when $k = 2p+1$ is odd, the second one is when $k = 2p$ is even.

In both cases, the weakly Hodge map $R_k : \mathcal{B}^0_1(M) \to \mathcal{B}^2_{k+1}(M)$ only has non-trivial pieces of Hodge bidegrees $(2,0)$, $(1,1)$ and $(0,2)$. Moreover, for any map Θ of odd degree we tautologically have $[\{\Theta, \Theta\}, \Theta] = 0$. Applying this to $\Theta = D_{\leq k}$ and collecting terms of augmentation degree k, we see that

$$C \circ R_k = 0.$$

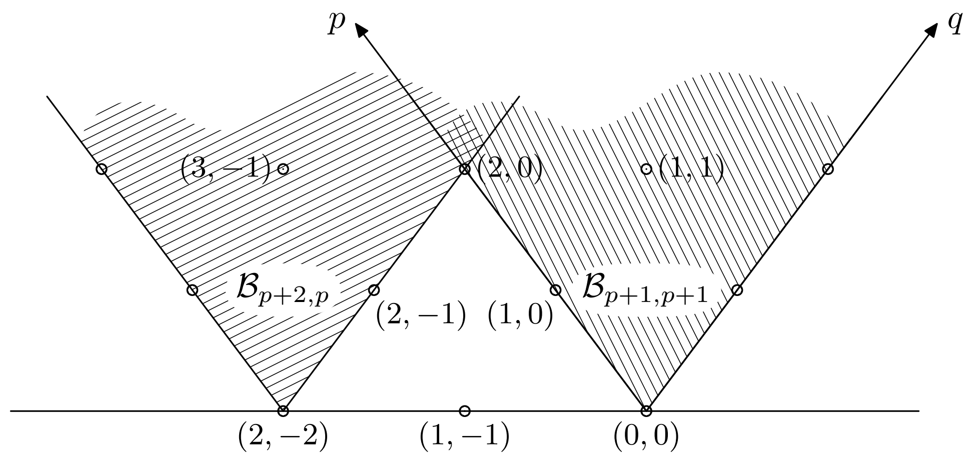

Figure 6.1: The augmentation bigrading on $\mathcal{B}^\bullet_{k+1}$ for an odd k, $k = 2p + 1$.

To track various components of the map $R_k : \mathcal{B}^0_1(M) \to \mathcal{B}^2_{k+1}(M)$ it is convenient to refine the augmentation grading on the Weil algebra $\mathcal{B}^\bullet(M)$ to an *augmentation bigrading* by setting

$$\deg S^{1,-1}(M) = \deg \Lambda^{1,0}(M) = (1, 0),$$
$$\deg S^{-1,1}(M) = \deg \Lambda^{0,1}(M) = (0, 1),$$

on the generator bundles $S^1(M, \mathbb{C}) = S^{1,-1}(M) \oplus S^{-1,1}(M)$ and $\Lambda^1(M, \mathbb{C}) = \Lambda^{1,0}(M) \oplus \Lambda^{0,1}(M)$. The augmentation bigrading will be denoted by lower indices, so that we have

$$\mathcal{B}^\bullet_k = \bigoplus_{p+q=k} \mathcal{B}^\bullet_{p,q}.$$

The relevant pieces of the augmentation bigrading on the bundle $\mathcal{B}^\bullet_{k+1}(M)$ are shown on Figure 6.1 for $k = 2p + 1$ odd, and on Figure 6.2 for $k = 2p$ even. The axes on the figures correspond to the grading by Hodge type. A Hodge bidegree component $\mathcal{B}^{p,q}_{m,n}$ can be non-trivial only when $p \geq m - n$ and $q \geq n - m$. Thus the component $\mathcal{B}^{\bullet,\bullet}_{m,n}$ is represented by an upward-looking angle with vertex $(m - n, n - m)$: a graded piece $\mathcal{B}^{p,q}_{m,n}$ can be non-trivial only if the point (p, q) lies in the interior (or on the boundary) of this angle.

Consider the Hodge bidegree decompositions

$$C = C^{1,0} + C^{0,1} \qquad\qquad \sigma = \sigma^{-1,0} + \sigma^{0,-1}$$

of the derivations C, σ of the Weil algebra $\mathcal{B}^\bullet(M)$. Then the augmentation bigrading is essentially the eigenvalue decomposition for the commutators

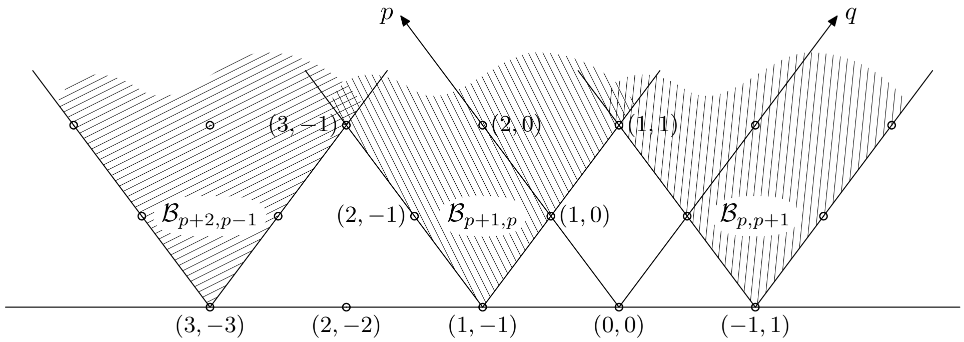

Figure 6.2: The augmentation bigrading on $\mathcal{B}^\bullet_{k+1}$ for an even k, $k = 2p$.

$\{C^{1,0}, \sigma^{-1,0}\}$ and $\{C^{0,1}, \sigma^{0,-1}\}$. More precisely, we have

$$\begin{aligned} &\{C^{1,0}, \sigma^{0,-1}\} = \{C^{0,1}, \sigma^{-1,0}\} = 0, \\ &\{C^{1,0}, \sigma^{-1,0}\} = m \,\mathsf{id} \ \text{ on } \mathcal{B}^\bullet_{m,n}, \\ &\{C^{0,1}, \sigma^{0,-1}\} = n \,\mathsf{id} \ \text{ on } \mathcal{B}^\bullet_{m,n} \end{aligned} \tag{6.2}$$

Indeed, since all these commutators are derivations of the Weil algebra, it suffices to check this on the generator bundles $S^1(M,\mathbb{C})$, $\Lambda^1(M,\mathbb{C})$, which is elementary. In particular, we see that both C and σ preserve the augmentation bidegree. The equalities (6.2) also immediately imply that

$$\{C, \sigma\} = k \,\mathsf{id} \ \text{ on } \mathcal{B}^\bullet_k.$$

One further corollary of (6.2) will be very important (we leave the proof to the reader as an easy exercise).

Lemma 6.1. *If $m, n \geq 1$, then the map $C^{1,0}$ is injective on every graded piece $\mathcal{B}^{p,q}_{m,n}$ with $q = n - m$, while $C^{0,1}$ is injective on $\mathcal{B}^{p,q}_{m,n}$ with $p = m - n$.*

Graphically, this means that $C^{1,0}$ is injective on $\mathcal{B}^{p,q}_{m,n}$ when the point (p,q) lies on the right-hand boundary of the angle representing $\mathcal{B}^{\bullet,\bullet}_{m,n}$, and $C^{0,1}$ is injective in this graded piece when the point (p,q) lies on the left-hand boundary of the same angle. We will call this *the boundary rule.*

We can now proceed with the proof of the induction step.

Case when $k = 2p+1$ is odd. Looking at Figure 6.1, we see that the only non-trivial augmentation-bidegree components of the map $R_k : \mathcal{B}^0_1(M) \to \mathcal{B}^2_{k+1}(M)$ are $R_{p,p+1}$ and $R_{p+1,p}$,

$$R_k = R_{p,p+1} + R_{p+1,p},$$

and the same is true for any weakly Hodge map $D_k : S^{1,-1}(M) \to \mathcal{B}^1_{k+1}(M)$,

$$D_k = D_{p,p+1} + D_{p+1,p}.$$

Moreover, on $S^{1,-1}(M) \subset \mathcal{B}^0(M)$ we have

$$\begin{aligned} D_k^{0,1} &= D_{p,p+1} & \qquad D_k^{1,0} &= D_{p+1,p}, \\ R_k^{0,2} &= R^{0,2}_{p,p+1} & \qquad D_k^{2,0} &= R^{2,0}_{p+1,p}, \end{aligned}$$

while $R_k^{1,1}$ further decomposes as $R^{1,1}_{p+1,p} + R^{1,1}_{p,p+1}$.

We have to find D_k which satisfies (6.1). In particular, we must have

$$R_k^{2,0} = -C^{1,0} \circ D_k^{1,0} \qquad\qquad R_k^{0,2} = -C^{0,1} \circ D_k^{0,1}. \tag{6.3}$$

But by the boundary rule the map $C^{1,0}$ is injective on $\mathcal{B}^{2,-1}_{p+2,p}$, while the map $C^{0,1}$ is injective on $\mathcal{B}^{1,0}_{p+1,p+1}$. Therefore there exists at most one weakly Hodge map $D_k : \mathcal{B}^0_{1,0}(M) \to \mathcal{B}^1_{k+1}$ which satisfies (6.3). Setting (again on $S^{1,-1}(M)$)

$$\begin{aligned} D_k^{0,1} &= D^{0,1}_{p,p+1} = -\frac{1}{p+1}\sigma^{0,-1} \circ R_k^{0,2}, \\ D_k^{1,0} &= D^{1,0}_{p+1,p} = -\frac{1}{p+2}\sigma^{-1,0} \circ R_k^{2,0}. \end{aligned}$$

gives this unique solution to (6.3). Indeed, we have

$$\begin{aligned} C^{1,0} \circ D_k^{1,0} &= -\frac{1}{p+2} C^{1,0} \circ \sigma^{-1,0} \circ R_k^{2,0} \\ &= -\frac{1}{p+2}\left(\{C^{1,0}, \sigma^{-1,0}\} \circ R^{2,0}_{p+1,p} + \sigma^{-1,0} \circ C^{1,0} \circ R_k^{2,0} \right). \end{aligned}$$

The second summand in the brackets vanishes since $C \circ R_k = 0$, while the first is equal to $(p+2)R_k^{2,0}$ by (6.2). This proves the first equation in (6.3). The second one is proved in exactly the same way.

It remains to prove that this map D_k satisfies not only (6.3) but also the stronger condition (6.1). To do this, note that

$$C \circ (C \circ D_k + R_k) = (C \circ C) \circ D_k + C \circ R_k = 0.$$

But from (6.3) we see that $C \circ D_k + R_k$ is of Hodge bidegree $(1,1)$. Therefore this implies that

$$C^{1,0} \circ (C \circ D_k + R_k) = C^{0,1} \circ (C \circ D_k + R_k) = 0.$$

The only possible non-trivial components of $C \circ D_k + R_k$ with respect to the augmentation bigrading have bidegrees $(p+1,p)$ and $(p,p+1)$, and by the boundary rule $C^{1,0}$ is injective on $\mathcal{B}^{2,0}_{p+1,p}$, while $C^{0,1}$ is injective on $\mathcal{B}^{2,0}_{p,p+1}$. Thus $C \circ D_k + R_k = 0$.

Case when $k = 2p$ is odd. Assume for the moment that $k \geq 4$, thus $p \geq 2$.

Looking at Figure 6.2, we see that *a priori* the map R_k can have three non-trivial augmentation-bidegree components, namely,

$$R_k = R^{0,2}_{p-1,p+1} + R_{p,p} + R^{2,0}_{p+1,p-1}.$$

However, since $C \circ R_k = 0$, and the map $C^{1,0}$ is injective on $\mathcal{B}^{3,-1}_{p+2,p-1}(M)$ by the boundary rule, we see that $R^{0,2}_{p+1,p-1} = 0$. Analogously, $R_{p-1,p+1} = 0$. Therefore R_k is of pure augmentation bidegree (p,p).

Since the map $D_k : S^1(M) \to \mathcal{B}^1_{k+1}(M)$ is weakly Hodge, it must also be of augmentation bidegree (p,p). Conversely, looking at the angle representing $\mathcal{B}^{\bullet,\bullet}_{p+1,p-1}$, we see that *every* real map $D_k : S^1(M,\mathbb{C}) \to \mathcal{B}^1_{k+1}$ of pure augmentation bidegree (p,p) is necessarily weakly Hodge. In particular, setting

$$D_k = -\frac{1}{k+1}\sigma \circ R_k \tag{6.4}$$

defines a weakly Hodge map. This map is a solution to (6.1):

$$C \circ D_k = -\frac{1}{k+1} C \circ \sigma \circ R_k = -\frac{1}{k+1}\{C,\sigma\} \circ R_k - \sigma \circ C \circ R_k = -R_k.$$

This solution is not unique. However, since k is even, we have the additional normalization condition $\sigma \circ D_k = 0$. This condition (automatically satisfied by the solution (6.4)) ensures uniqueness. Indeed, the difference $P = D_k - D'_k$ between two solutions D_k, D'_k must satisfy $C \circ P = \sigma \circ P = 0$, which implies

$$P = \frac{1}{k+1}\{C,\sigma\} \circ P = 0.$$

Finally, it remains to consider the case $k = 2$. The general argument works in this case just as well, with a single exception. Since $p-1 = 0$ is no longer strictly positive, the boundary rule does not apply: it is not true that $C^{1,0}$ is injective on $\mathcal{B}^{3,-1}_{p+1,p-1} = \mathcal{B}^{3,-1}_{3,0}$ (in fact, on this graded piece $C^{1,0}$ is equal to zero). Therefore the component $R_{p+1,p-1}$ does not vanish automatically. However, this component

$$R^{2,0}_{2,0} : S^{1,-1}(M) \to \mathcal{B}^{3,-1}_{3,0}(M) \cong \Lambda^{2,0}(M) \otimes S^{1,-1}(M)$$

is precisely the $(2,0)$-curvature of the connection ∇ on M. It vanishes by the second assumption on this connection. $\square$

7 Metrics.

The last Section essentially finishes the proof of the hypercomplex Theorem 1.4 (it remains to prove Lemma 5.2). We will now sketch a proof of the hyperkähler Theorem 1.1.

As we have already noted, Theorem 1.1 will be a corollary of Theorem 1.4. Namely, given a Kähler manifold M we proceed in the following way. First we note that the Levi-Civita connection ∇_{LC} on M has no torsion and no $(2,0)$-curvature. Therefore Theorem 1.4 applies to ∇_{LC} and provides a hypercomplex structure on the total space $X = \overline{T}M$. Then we show that every Hermitian metric on M which is preserved by ∇_{LC} (in particular, the given Kähler metric) extends uniquely from the zero section $M \subset \overline{T}M$ to a (formal germ of a) hyper-hermitian metric on the hypercomplex manifold $X = \overline{T}M$ which is compatible with the hypercomplex structure. After this, we finish the proof by identifying the holomorphically symplectic manifolds $\overline{T}M$ and T^*M.

We will go through these steps in reverse order, starting with the last one.

Lemma 7.1. *Assume given a hypercomplex structure on the total space $X = \overline{T}M$ which satisfies the conditions of Theorem 1.4. Let h be a $U(1)$-invariant hyperkähler metric on X compatible with this hypercomplex structure, and let $\Omega_X \in \Lambda^{2,0}(X)$ be the associated holomorphic 2-form. Let T^*M be the total space of the cotangent bundle to M equipped with the standard holomorphic 2-form Ω.*

Then there exists a unique $U(1)$-equivariant biholomorphic map $\eta : X \to T^*M$ such that $\Omega_X = \eta^*\Omega$.

Proof. Since the map η must be $U(1)$-equivariant, it must commute with the canonical projections $\rho : X, T^*M \to M$ and send the zero section $M \subset X$ to the zero section $M \subset T^*M$. Denote by ϕ the differential of the $U(1)$-action. Then we also must have

$$\Omega_X \lrcorner \phi = \eta^*\Omega \lrcorner \phi = \eta^*(\Omega \lrcorner \phi).$$

But the 1-form $\alpha = \Omega \lrcorner \phi$ is the tautological 1-form $\alpha \in \rho^*(\Lambda^1(M)) \subset \Lambda^1(T^*M)$. Therefore the 1-form $\rho^*\alpha$ on X completely defines the map η.

Conversely, the form $\alpha_X = \Omega_X \lrcorner \phi$ satisfies $\alpha_X = \eta^*\alpha$ for a unique map $\eta : X \to T^*M$. Since the metric h is $U(1)$-invariant, the forms Ω_X and α_X are of weight 1. Therefore the map $\eta : X \to T^*M$ is $U(1)$-equivariant. By the Cartan homotopy formula, we have

$$\Omega_X = d\alpha_X = d\eta^*\alpha = \eta^* d\alpha = d\Omega. \qquad \square$$

We will now explain how to construct the metric h – or, equivalently, the associated holomorphic 2-form $\Omega_J \in \Lambda^{2,0}(X)$.

Keep the notation of last two Sections. Let $\omega \in \Lambda^{1,1}(M, \mathbb{C})$ be the Kähler form (more generally, any $(1,1)$-form preserved by the connection). We have to prove that there exists a unique (formal germ of a) holomorphic $(2,0)$-form $\Omega \in \Lambda^{2,0}_J(X)$ which is of H-type $(1,1)$ and whose restriction to the zero section $M \subset X$ coincides with ω (since the positivity (3.12) is an open condition, it is satisfied automatically in a neighborhood of the zero section $M \subset X$).

To reformulate this in terms of M, consider the complex $\Lambda^{2,\bullet}_J(X)$ of Hodge bundles on $\overline{T}M$ with the Dolbeault differential $D = \bar\partial_J : \Lambda^{2,\bullet}_J(X) \to \Lambda^{2,\bullet+1}(X)$. Denote by

$$\mathcal{C}^{\bullet+2}(M) = \rho_*\Lambda^{2,\bullet}_J(X)$$

its direct image on M (the grading is shifted by 2 to make it compatible with the Hodge degrees). We are given a section $\omega \in \Lambda^{1,1}(M)$ of Hodge type $(1,1)$. We have to prove that there exists a section $\Omega = \Omega_J \in \mathcal{C}^{1,1}(M)$ such that $\Omega = \omega$ on the zero section $M \subset X$ and $D\Omega = 0$.

Since $\Lambda^{2,\bullet}_J(X) \cong \Lambda^{2,0}_J(X) \otimes \Lambda^{0,\bullet}_J(X)$ and $\Lambda^{2,0}_J(X) \cong \rho^*\Lambda^2(M, \mathbb{C})$, the complex $\mathcal{C}^\bullet(M)$ is a free module

$$\mathcal{C}^\bullet(M) \cong L^2(M) \otimes \mathcal{B}^\bullet(M)$$

over the Weil algebra $\mathcal{B}^\bullet(M) = \rho_*\Lambda^{0,\bullet}_J(X)$ generated by some subbundle

$$L^2(M) \subset \mathcal{C}^2(M)$$

which is isomorphic to $\Lambda^2(M, \mathbb{C})$. We introduce the augmentation grading on the $\mathcal{B}^\bullet(M)$-module $\mathcal{C}^\bullet(M)$ by setting $\deg L^2(M) = 2$. Just as in Proposition 5.3, the proof will proceed by induction on the augmentation degree – namely, we will construct the form $\Omega \in \mathcal{C}^{1,1}(M)$ as a sum

$$\Omega = \Omega_0 + \dots + \Omega_k + \dots$$

with $\Omega_k \in \mathcal{C}^{1,1}_k(M)$ of augmentation degree $k+2$. We begin with the induction step. It is completely parallel to Proposition 5.3, so we give only a sketch.

Induction step – a sketch. We can assume that we already have

$$\Omega_{<k} = \Omega_0 + \cdots + \Omega_{k-1}$$

such that $\Phi = D\Omega_{<k}$ is of augmentation degree $\geq k+2$. Denote by $\Phi_k = \Phi_k^{2,1} + \Phi_k^{1,2}$ the component of augmentation degree exactly $k+2$. We have to show that there exists a unique $\Omega_k \in \mathcal{C}^{1,1}_{k+2}$ such that $\Phi_k = D_0\Omega_k$.

The derivations C and σ of the Weil algebra $\mathcal{B}^\bullet(M)$ extend to endomorphisms of the free module $L^2 \otimes \mathcal{B}^\bullet(M)$ by setting $C = \sigma = 0$ on $L^2(M)$. Just as on $\mathcal{B}^\bullet(M)$, we have $D_0 = C$. For every $k \geq 0$, we have $\{C, \sigma\} = k\,\mathsf{id}$ on $\mathcal{C}^\bullet_{k+2}$. This immediately implies that $C = D_0$ is injective on $\mathcal{C}^{1,1}_{k+2}(M)$ for $k \geq 1$, which proves the uniqueness of Ω_k.

The space $\mathcal{C}^\bullet_{k+2}$ splits into the sum of parts of the form

$$L^{p,q} \otimes \mathcal{B}^\bullet_{m,n}, \qquad p+q=2;\ m+n=k;\ p,q,m,n \geq 0.$$

Such a part can have a non-trivial piece of Hodge bidegree p_1, q_1 only if $p_1 \geq p+m-n$ and $q_1 \geq q+n-m$. Having in mind the graphical representation as in Figure 6.1 and Figure 6.2, we will say that the part $L^{p,q} \otimes \mathcal{B}^\bullet_{m,n} \subset \mathcal{C}^\bullet_{k+2}$ is *an angle based at* $(p+m-n, q+n-m)$. Each angle is preserved by the maps C and σ. In this terminology, $\mathcal{C}^{2,1}_{k+2}$ can intersect non-trivially with various angles based at $(2,0)$ and $(1,1)$, while $\mathcal{C}^{1,2}_{k+2}$ can intersect non-trivially with angles based at $(1,1)$ and $(0,2)$. But by induction we have $C\Phi_k = 0$, which implies that $C^{1,0}\Phi_k^{2,1} = C^{0,1}\Phi_k^{1,2} = 0$. Applying the boundary rule Lemma 6.1, we see that both $\Phi_k^{2,1} \in \mathcal{C}^{2,1}_{k+2}$ and $\Phi_k^{1,2} \in \mathcal{C}^{1,2}_{k+2}$ must lie entirely within angles based at $(1,1)$. Therefore $\sigma\Phi_k$ must also lie within angles based in $(1,1)$. This means that $\sigma\Phi_k \in \mathbb{C}^{1,1}_{k+2}$ is of Hodge type $(1,1)$, and we can set $\Omega_k = \frac{1}{k}\sigma\Phi_k$. □

We now have to base the induction – namely, to find the section $\Omega_0 \in \mathcal{C}^{1,1}_2(M)$ with correct restriction to the zero section $M \subset X$, and to handle those angles $L^{p,q}(M) \otimes \mathcal{B}^\bullet_{m,n}$ to which the boundary rule does not apply – which means the angles with $m = 0$ or $n = 0$. There are three such angles. We denote the corresponding components of the derivation $D : L^{1,1}(M) \to L^2(M) \otimes \mathcal{B}^1(M)$ by

(7.1) $$D_1 : L^{1,1}(M) \to L^{1,1}(M) \otimes \mathcal{B}^1_1(M),$$

(7.2) $$D_2^l : L^{1,1}(M) \to L^{2,0}(M) \otimes \mathcal{B}^{-1,2}_{0,2}(M),$$

(7.3) $$D_2^l : L^{1,1}(M) \to L^{0,2}(M) \otimes \mathcal{B}^{2,-1}_{2,0}(M).$$

We have to choose Ω_0 so that $D_1\Omega_0 = D_2^l\Omega_0 = D_2^r\Omega_0 = 0$. We note that $D_2^l\Omega_0 = 0$ implies $D_2^r\Omega_0 = 0$ by complex conjugation.

Moreover, we note that we have one more degree of freedom. So far nothing depended on the choice of the generator subbundle $L^2(M) \subset \mathcal{C}^2(M)$ – all we needed was to know that it exists. We will now make this choice. It will not be the most obvious one, but the one which will make computations as easy as possible. We consider the splitting

$$\Lambda^1(X,\mathbb{C}) = \Lambda^{1,0}_J(X) \oplus \rho^*\Lambda^1(M,\mathbb{C}) \tag{7.4}$$

given by the Hodge connection $\Theta : \Lambda^1(X,\mathbb{C}) \to \rho^*\Lambda^1(M,\mathbb{C})$. This splitting induces a bigrading on the de Rham algebra $\Lambda^\bullet(X,\mathbb{C})$. The complex $\Lambda^\bullet(X,\mathbb{C})$ with this bigrading is a bicomplex which we will denote by $\Lambda^{\bullet,\bullet}_\Theta(X)$. More precisely, the de Rham differential d is a sum of two anti-commuting differentials

$$\widetilde{d} : \Lambda^{\bullet,\bullet}_\Theta(X) \to \Lambda^{\bullet+1,\bullet}_\Theta(X) \qquad\qquad D : \Lambda^{\bullet,\bullet}_\Theta(X) \to \Lambda^{\bullet,\bullet+1}_\Theta(X)$$

(this is essentially equivalent to the flatness of the Hodge connection Θ).

Since the first term in the splitting (7.4) is $\Lambda^{1,0}_J(X)$, the subcomplexes $\Lambda^{\geq k,\bullet}_\Theta(X)$ and $\Lambda^{\geq k,\bullet}_J(X)$ of the de Rham complex $\Lambda^\bullet(X,\mathbb{C})$ are the same for every k. Therefore the associated graded quotients $\Lambda^{k,\bullet}_\Theta(X)$ and $\Lambda^{k,\bullet}_J(X)$ are also isomorphic for every k. In particular, we have

$$\mathcal{C}^{\bullet+2}(M) = \rho_*\Lambda^{2,0}_J(X) \cong \rho_*\Lambda^{2,\bullet}_\Theta(X).$$

On the other hand, since

$$\Lambda^{1,0}_\Theta(X) = \Lambda^1(X,\mathbb{C})/\rho^*\Lambda^1(M,\mathbb{C}) \cong \Lambda^1(X/M,\mathbb{C})$$

is the bundle of relative 1-forms on X/M, the quotient complex $\Lambda^{\bullet,0}_\Theta(X)$ with the differential $\widetilde{d}$ is canonically isomorphic

$$\Lambda^{\bullet,0}_\Theta(X) \cong \Lambda^\bullet(X/M,\mathbb{C}) \tag{7.5}$$

to the relative de Rham complex $\Lambda^\bullet(X/M,\mathbb{C})$. We use this identification and choose as

$$L^k(M) \subset \rho_*\Lambda^\bullet(X/M,\mathbb{C}) \cong \rho_*\Lambda^{\bullet,0}_\Theta(X)$$

the subbundle of k-forms which are constant along the fibers of the projection $\rho : X \to M$ (by a constant k-form on a vector space V with a basis $e_1,\ldots,e_n$ we mean a linear combination of forms $e_{a_1}\wedge\cdots\wedge e_{a_k}$ with constant coefficients).

This choice guarantees that the relative de Rham differential $\widetilde{d} : L^{\bullet}(M) \otimes \mathcal{B}^0(M) \to L^{\bullet+1}(M) \otimes \mathcal{B}^0(M)$ takes a very simple form. Namely, it vanishes on $L^k(M)$, and induces an isomorphism

$$\widetilde{d} : S^1(M) \cong L^1(M) \tag{7.6}$$

between the generator subbundles $S^1(M) \subset \mathcal{B}^0(M)$ and $L^1(M) \subset L^1(M) \otimes \mathcal{B}^1(M)$. This is important because by construction $\widetilde{d}$ anti-commutes with D. Moreover, since $\widetilde{d}$ obviously preserves the augmentation degrees, it anti-commutes separately with each of the components D_k. Since we already know the derivations D_k on $\mathcal{B}^{\bullet}(M)$, the isomorphism (7.6) will allow us to compute individual components $D_k : L^{\bullet}(M) \to L^{\bullet}(M) \otimes \mathcal{B}^1_k(M)$.

The first result is the following: the map $D_1 = D_1^{1,1}$ in (7.1) is minus the connection

$$\nabla : L^{1,1}(M) \to L^{1,1}(M) \otimes \Lambda^1(M, \mathbb{C})$$

on the bundle $L^{1,1}(M) \cong \Lambda^{1,1}(M, \mathbb{C})$. Indeed, $D_1 : L^{\bullet}(M) \to L^{\bullet}(M) \otimes \Lambda^1(M, \mathbb{C})$ is a derivation of the exterior algebra $L^{\bullet}(M)$. Thus it suffices to prove that $D_1 = -\nabla$ on $L^1(M)$. Since $\widetilde{d}D_1 = -D_1\widetilde{d}$, this follows from (7.6) and the construction of the map $D : \mathcal{B}^{\bullet}(M) \to \mathcal{B}^{\bullet+1}(M)$ given in Section 6. This shows that taking

$$\Omega_0 = \omega \in L^{1,1}(M) \cong \Lambda^{1,1}(M) \tag{7.7}$$

guarantees that $D_1\Omega_0 = 0$.

At this point we will also choose the isomorphism $L^k(M) \cong \Lambda^1(M, \mathbb{C})$ – namely, we take the composition of the embedding $L^k(M) \subset \Lambda^k(X/M, \mathbb{C}) \cong \Lambda^{k,0}_{\Theta}(X)$ and the restriction $i^*\Lambda^{k,0}(X) \to \Lambda^k(M, \mathbb{C})$ to the zero section $i : M \hookrightarrow X$. Then the form Ω_0 defined by (7.7) automatically restricts to ω.

It remains to prove that $D_2^l\Omega_0 = 0$. This is a corollary of the following claim.

Lemma 7.2. *The map D_2^l defined in* (7.3) *is the composition of the curvature*

$$R : L^{1,1}(M) \to L^{1,1}(M) \otimes \Lambda^{1,1}(M)$$

of the connection $\nabla = D_1 : L^{1,1}(M) \to L^{1,1}(M) \otimes \Lambda^1(M, \mathbb{C})$ and a certain bundle map

$$Q : L^{1,1}(M) \otimes \Lambda^{1,1}(M) \to L^{0,2}(M) \otimes \mathcal{B}^{2,-1}_{2,0}(M).$$

Proof. Extend the map D_2^l to an algebra derivation

$$D_2^l : L^\bullet(M) \to L^\bullet(M) \otimes \mathcal{B}_{2,0}^{2,-1}(M) \cong L^\bullet(M) \otimes S^{1,-1}(M) \otimes \Lambda^{1,0}(M)$$

by setting $D_2^l = 0$ on $L^{0,1}(M)$ and taking as

$$D_2^l : L^{1,0}(M) \to L^{0,1}(M) \otimes \mathcal{B}_{2,0}^{2,-1}(M)$$

the corresponding component of the map $D_2 : L^{1,0}(M) \to L^1(M) \otimes \mathcal{B}_2^1(M)$. Since D_2^l is a derivation of the algebra $L^\bullet(M)$ which vanishes on $L^{0,1}(M)$, on $L^{1,1}(M)$ it is equal to the composition

$$\begin{aligned} L^{1,0}(M) \otimes L^{0,1}(M) &\xrightarrow{D_2^l \otimes \mathsf{id}} \mathcal{B}_{2,0}^{2,-1}(M) \otimes L^{0,1}(M) \otimes L^{0,1}(M) \longrightarrow \\ &\xrightarrow{\mathsf{id} \otimes \mathsf{Alt}} \quad \mathcal{B}_{2,0}^{2,-1}(M) \otimes L^{0,2}(M) \end{aligned}$$

(here $\mathsf{Alt} : L^{0,1}(M) \otimes L^{0,1}(M) \to \Lambda^{0,2}(M)$ is the alternation map). Therefore it suffices to prove that on $L^{1,0}(M)$ we have

$$D_2^l = P \circ R : L^{1,0}(M) \to L^{1,0}(M) \otimes \Lambda^{1,1}(M) \to L^{0,1} \otimes \mathcal{B}_{2,0}^{2,-1}(M)$$

for a certain bundle map $P : L^{1,0}(M) \otimes \Lambda^{1,1}(M) \to L^{0,1} \otimes \mathcal{B}_{2,0}^{2,-1}(M)$. Since $D_2 \widetilde{D} = -\widetilde{d} D_2$, this follows directly from (7.6) and (6.4) with $k = 2$. $\square$

This Lemma implies that $D_2^l(\Omega_0) = Q(R(\omega)) = Q(\nabla(\nabla(\omega))) = 0$. This finishes the proof of Theorem 1.1.

The last application of the formalism that we have developed in this Section is the proof of Lemma 5.2.

Proof of Lemma 5.2. The Obata connection ∇_O on a hypercomplex manifold X is defined as follows: for every $(0,1)$-form $\alpha \in \Lambda^{0,1}(X)$ the $(1,0)$-part $\nabla_O^{1,0}$ is equal to

$$\nabla_O^{1,0} \alpha = \partial \alpha \in \Lambda^{0,1}(M) \otimes \Lambda^{1,0}(M),$$

while the $(0,1)$-part satisfies

$$\nabla_O^{0,1} \alpha = -(\mathsf{id} \otimes j)(\partial(j(\alpha))) \in \Lambda^{0,1}(M) \otimes \Lambda^{0,1}(M).$$

The first condition in fact automatically follows from the absence of torsion.

Consider a hypercomplex structure on $X = \overline{T}M$ corresponding to a torsion-free connection ∇ on M, and let $\alpha \in \Lambda^{0,1}(M)$ be a $(0,1)$-form on M. We have to prove that $\nabla_O^{0,1}(\rho^*\alpha) = \rho^* \nabla^{0,1} \alpha$ on the zero section $i : M \hookrightarrow X$. It suffices to prove that

$$i^* \bar{\partial} j(\rho^* \alpha) = i^* (\mathsf{id} \otimes j) \left(\nabla^{0,1} \alpha \right) \tag{7.8}$$

as section of the bundle $\Lambda^{0,1}(M) \otimes i^*\Lambda^1(X, \mathbb{C})$ on the zero section $M \subset X$. The canonical isomorphism $H : \Lambda^{\bullet,\bullet}(X) \cong \Lambda^{\bullet,\bullet}_J(X)$ sends the Dolbeault differential $\bar\partial$ to the component $D^{1,0}$ of the Dolbeault differential $D = \bar\partial_J : \Lambda^{\bullet,\bullet}_J(X) \to \Lambda^{\bullet,\bullet+1}_J(X)$. Moreover, the composition of the map $H \circ j : \rho^*\Lambda^{0,1}(M) \to \Lambda^{1,0}_J(X)$ and the restriction to the zero section $M \subset X$ induces a bundle map

$$P : \Lambda^{0,1}(M) \to L^1(M) = i^*\Lambda^{1,0}_J(X) \subset i^*\Lambda^1(X, \mathbb{C}).$$

It is easy to check that this map is proportional to the canonical embedding $\Lambda^{0,1}(M) \cong L^{0,1}(M) \hookrightarrow L^1(M)$. Therefore it commutes with the connection $\nabla^{1,0}$ – namely, we have $(\mathsf{id} \otimes P) \circ \nabla^{1,0} = \nabla^{1,0} \circ P$. The equation (7.8) becomes

$$D^{1,0}_1 P(\alpha) = \nabla^{1,0} P(\alpha).$$

But we have already proved that on $L^1(M)$ we have $\nabla = D_1$. $\square$

8 Symmetric spaces.

To illustrate our rather abstract methods by a concrete example, we would like now to derive a formula for the canonical hypercomplex structure on $X = \overline{T}M$ in the case when M is a symmetric space. In this case the classic equality $\nabla R = 0$ bring many simplifications, so that the constructions of Section 6 can be seen through to a reasonably explicit final result. The formula that we obtain is similar to the one obtained by O. Biquard and P. Gauduchon in [BG].

Let us introduce some notations. Let M be a symmetric space with Levi-Civita connection ∇ and curvature R. Consider the total space $X = \overline{T}M$ with the canonical projection $\rho : \overline{T}M \to M$. Let $A : \rho^*\Lambda^1(M) \to \rho^*\Lambda^1(M)$ be the endomorphism of the pullback bundle $\rho^*\Lambda^1(M)$ given by

$$A(\xi)(\alpha) = -\frac{1}{3} R_m(\alpha) \lrcorner (\xi \otimes \xi), \qquad \alpha \in \rho^*\Lambda^1(M). \tag{8.1}$$

Here (ξ, m), $m \in M$, $\xi \in T_mM$ is a point in $\overline{T}M$, R_m is the curvature evaluated at the point $m \in M$, and $R(\alpha)$ is interpreted as a section of the bundle $\rho^*\Lambda^1(M) \otimes \rho^*\Lambda^{1,1}(M)$. Let also

$$f(z) = \sum_{p \geq 1} f_p z^p$$

be the generating function for the recurrence relation

$$f_p = -\frac{1}{2p+1} \sum_{1 \leq l \leq p-1} f_l f_{p-l} \tag{8.2}$$

with the initial condition $f_1 = 1$. In other words, $f(z)$ is the solution of the ODE

$$2zf'(z) + f(z) + f^2(z) = 3z$$

with the initial condition $f(0) = 0$. With these notations, we can formulate our result.

Proposition 8.1. *Let $I : \Lambda^1(M) \to \Lambda^1(M)$ be the complex structure operator on M. Then the map $J : \Lambda^1(X) \to \Lambda^1(X)$ for the canonical normalized hypercomplex structure on X is given by the matrix*

$$J = \begin{pmatrix} 0 & f(A)I \\ I(f(A))^{-1} & 0 \end{pmatrix}$$

with respect to the decomposition

$$\Lambda^1(X) = \Lambda^1(X/M) \oplus \rho^*\Lambda^1(M) \cong \rho^*\Lambda^1(M) \oplus \rho^*\Lambda^1(M)$$

associated to the Levi-Civita connection on M.

We have already noted that this result is very similar to the formula obtained in [BG]. However, it is not the same. The reason is the following: Biquard and Gauduchon work with the cotangent bundle T^*M, and they use the normalization natural to the cotangent bundle. As the result, their analog of the map A is slightly different, and the function $f(z)$ is also different – in particular, it is given by an explicit expression. To compare our results with those of [BG], one should either compute the normalization map $\mathcal{L} : T^*M \to \overline{T}M$ for the Biquard-Gauduchon hypercomplex structure, or go the other way around and compute the map $\eta : \overline{T}M \to T^*M$ provided by Lemma 7.1. Unfortunately, I have not been able to do either.

Proof of Proposition 8.1. Throughout the proof, we will freely use the notation of preceding Sections.

The main simplification in the case of symmetric spaces is the well-known equality $\nabla R = 0$. For our construction it immediately implies that the odd augmentation degree components D_{2p+1} of the weakly Hodge derivation $D : \mathcal{B}^{\bullet}(M) \to \mathcal{B}^{\bullet+1}(M)$ vanish for $p \geq 1$. The only non-trivial component is D_1. By (4.4), this immediately shows that the connection Θ_{Re} on the

fibration $\rho : X \to M$ is simply the linear connection associated to the Levi-Civita connection ∇. Applying Lemma 4.3, we see that all we have to prove is the equality $R_J = f(A)$. By Lemma 4.4 this can be rewritten in terms of the Weil algebra $\mathcal{B}^\bullet(M)$ as

$$D_+ \circ \sigma = f(A) : \Lambda^1(M, \mathbb{C}) \to \mathcal{B}^1(M).$$

Moreover, by (6.4) with $k = 2$ the endomorphism A of the $\mathcal{B}^0(M)$-module $\mathcal{B}^1(M) \cong \rho_* \rho^* \Lambda^1(M, \mathbb{C})$ satisfies

$$A = D_2 \circ \sigma$$

on $\Lambda^1(M, \mathbb{C})$. In fact, this should be taken as the definition – I apologize to the reader for any possible mistakes in writing down the explicit formula (8.1).

Next we note that since D_2 vanishes on $\Lambda^1(M, \mathbb{C})$, the endomorphism A is in fact equal

$$A = \{D_2, \sigma\}$$

to the commutator $\{D_2, \sigma\}$. Moreover, this formula holds not only on the generator subbundle $\Lambda^1(M, \mathbb{C}) \subset \mathcal{B}^1(M)$, but on the whole $\mathcal{B}^0(M)$-module $\mathcal{B}^1(M)$. Indeed, by the normalization condition (5.2) this commutator vanishes on the generator subbundle $S^1(M) \subset \mathcal{B}^0(M)$. Since it is a derivation of the Weil algebra, it vanishes on the whole $\mathcal{B}^0(M)$. Therefore it restricts to a map of $\mathcal{B}^0(M)$-modules on $\mathcal{B}^1(M) \subset \mathcal{B}^1(M)$.

We now trace one-by-one the induction steps in the proof of Proposition 5.3. Since all the odd terms vanish, we only need to consider the even terms D_{2p}. We have to prove that

$$\{D_{2p}, \sigma\} = f_k A^k : \mathcal{B}^1(M) \to \mathcal{B}^1(M).$$

Since both sides are maps of $\mathcal{B}^0(M)$-modules, it suffices to prove this on $\Lambda^1(M, \mathbb{C}) \subset \mathcal{B}^1(M)$, where we can replace $\{D_{2p}, \sigma\}$ with $D_p \circ \sigma$. By (6.4), on $S^1(M) \subset \mathcal{B}^0(M)$ we have

$$D_{2p} = -\frac{1}{2p+1} \sum_{1 \leq l \leq p-1} \sigma \circ D_{2l} \circ D_{2(p-l)}.$$

Comparing this to (8.2), we see that it suffices to prove that

$$\sigma \circ D_m \circ D_n \circ \sigma = \{D_m, \sigma\} \circ \{D_n, \sigma\} : \Lambda^1(M) \to \mathcal{B}^1(M)$$

for all even $m, n \geq 2$. Writing out the commutators on the right-hand side, we see that the difference is equal to

$$D_m \circ \sigma \circ D_n \circ \sigma + D_m \circ \sigma \circ \sigma \circ D_n + \sigma \circ D_m \circ \sigma \circ D_n.$$

The first summand vanishes since by the normalization condition (5.2) we have $\sigma \circ D_n = 0$ on $S^1(M) = \sigma(\Lambda^1(M, \mathbb{C}))$. The last two summands vanish since $D_n = 0$ on $\Lambda^1(M, \mathbb{C})$. $\square$

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